

Fluid Dynamics (3) - MEP 303A

By Dr. Mohsen soliman

Course Specifications

A- Basic Information

Title: Fluid Dynamics (3)

Code: 303A

Credit Hours: 4 hrs

Lecture: 3 hrs/wk, Tutorial: 2 hrs/wk, Practices:--, Total: 5 hrs/wk

B- Professional Information

1- Overall Aims of Course

This course builds upon the 2nd year basic material in “Fluid Mechanics” but is more applied in nature. The course deals with different types of fluid flow problems from the differential control volume analysis point of view. The course examines various types of flows such as internal viscous flow, frictionless flow and boundary layer flow.

2- Intended Learning Outcomes of Course (ILOs)

a- Knowledge and Understanding

- a1- Introducing Governing conservation equations (mass & linear momentum) for a differential control volume.
- a2- Examining physical meaning and weight of terms in the above equations.
- a3- Evaluating the role of viscous effects in different fluid flows.
- a4- Introducing the concept of frictionless flow.
- a5- Examining the differential equations of the boundary layer flow.

b- Intellectual Skills:

- b1- **Analysis:** Selecting the differential equations in various coordinate-systems (cartisine, cylindrical and spherical forms) suitable for the flow examined.
- b2- **Creative thinking:** Introducing the concepts of superposition, images and conformal-mapping of elementary plane flows in studying frictionless flow to simulate some practical engineering problems.
- b3- **Problem Solving:** Introducing concept of solution batching to link frictionless flow region to boundary layer viscous flow region in the real flow field problems.

c- Professional and Practical Skills:

- c1- Applying fluid dynamics to deal with some important engineering problems such as lubrication in bearings, drag and lift on immersed bodies or airfoil shapes.
- c2- **Drawing:** Sketching and presenting some flow patterns (stream and potential lines) around immersed bodies or airfoil shapes.
- c3- **Computing:** Introducing different types of numerical solution techniques to solve differential equations such as using stream function method for frictionless flow and Blasius Solution for the boundary layer flow analysis.

d-General and Transferable Skills

- d1- **Engineering design:** Building ability to identify practical problems and select between differential or integral control volume analysis to solve them.
- d2- **Building ability to compare** between analytical or numerical methods.
- d3- **Use of technological tool:** (e.g., the Internet) to prepare Report assignments.
- d4- Practicing using numerical solution tables (e.g., Blasius Solution Table).
- d5- Working in group.

3- Contents

Topic	No. of hrs	Lecture	Tutorial
Chapter 1: Differential equation of mass conservation	4	2	2
Driving Navier-Stokes equations (linear momentum) for Newtonian fluids, angular momentum and energy eqns.	6	4	2
Viscous flow in pipes and ducts	4	2	2
Flow between parallel plates with pressure gradients	6	4	2
Chapter 2: Differential equations for frictionless flow (Euler's eqns.)	4	2	2
Stream and potential functions, vorticity, irrotationality, elementary plane-flow solutions.	6	4	2
Superposition of plane-flows and Images	4	2	2
Plane flows past closed body shapes, axi-symmetric flows	6	4	2
Lift & drag on submerged bodies in ideal flow, airfoil theory	4	2	2
Chapter 3: Introduction to Boundary Layer flows, the differential equations, Exact equations for 2-D flow	6	4	2
Blasius exact solution for laminar flow, the Momentum Integral equations	4	2	2
Approximate solutions for 2-D laminar and turbulent boundary layers	6	4	2
Thermal Boundary Layer over a flat plate	4	2	2
Revision of the course to confirm the objectives	6	4	2
Time for Preparing for the term exam	5	3	2
Total teaching hours in 15 weeks (+ 1 office hr/wk)	75	45	30

4- Teaching and Learning Methods

- 4.1- Lectures and problem solving in tutorial classes.
- 4.2- Information collection from text material, class notes and the Internet sites.
- 4.3- Report and research assignments. Three assignment Sheets (1, 2 and 3)
- 4.4- Group discussions in lectures and tutorial classes.
- 4.5- Hand-outs materials.

5- Student Assessment Methods

- 5.1- Test (1) & Report (1) to assess understanding Chapter (1) and solving Sheet #1
- 5.2- Test (2) & Report (2) to assess understanding Chapter (2) and solving Sheet #2
- 5.3- Mid-term exam to assess understanding Chapters (1-3) and solving Sheets #1-3
- 5.4- Final Term Exam to assess gains of all completed topics.

Assessment Schedule

Test (1) & Report (1)	End of Week 4
Test (2) & Report (2)	End of Week 9
Mid-term Exam	End of Week 11
Final Term Exam	End of Term

Weighting of Assessments

Tests 1, 2, assignments & class performance	10 %
Reports 1, 2	10 %
Mid-term Exam	20 %
Final-term Examination	60 %
<hr/>	
Total	100 %

6- List of References

6.1- Course Notes: Compiled Notes corresponding to different course sections

6.2- Essential Books (Text Books):

B.R. Munson, D. F. Young, and T. H. Okishi, “*Fundamentals of Fluid Mechanics*”, John Wiley & Sons, Inc., New York, 4th Edition (2002).

6.3- Recommended Books:

Frank M. White “*Fluid Mechanics*”, 2nd ed., McGraw Hill, 1986.

R.W.Fox & A.T.McDonald “*Introd. to Fluid Mechanics*”, 3rd ed., John Wiley & Sons, 1989

6.4- Hand-outs and Web Sites information,... etc

7- Facilities Required for Teaching and Learning

Data Show, Screen, new reference in library

Enhancing the ability to think for students in Engineering Schools

Course Coordinator: Ass. Prof. Mohsen S. Mohamed

Date: 20/9/2017

Table of Contents of Part (1)

Sec.		page
1.0	Introduction	6
1.1	Fluid Element Kinematics	8
1.1.1	Velocity and Acceleration Fields Revisited	8
1.1.2	Linear Motion and Deformation	9
1.1.3	Angular Motion and Deformation	11
1.2	Conservation of Mass	14
1.2.1	Differential Form of Continuity Equation	14
1.2.2	Cylindrical Polar Coordinates	18
1.2.3	The Stream Function	19
1.3	Conservation of Linear Momentum	22
1.3.1	Description of Forces Acting on Differential Element	23
1.3.2	Equations of Motion	25
1.4	Inviscid Flow	26
1.4.1	Euler's Equations of Motion	27
1.4.2	The Bernoulli Equation	27
	(sections 1.5 – 1.7 are omitted)	
1.8	Viscous Flow	29
1.8.1	Stress Deformation Relationships	29
1.8.2	The Navier-Stokes Equations	30
1.9		
1.9.1	The Differential Equation of Angular Momentum	34
1.9.2	The Differential Equation of Energy	35
1.9.4	Boundary Conditions for the Basic Differential Equations	37
1.9.5	Summary of Equations of Motion and Energy in Cylindrical Coordinates	41
1.10	Other Aspects of Differential Analysis	42
1.10.1	Numerical Methods	43
	1st Fluid Report	47
	Word Problems	48
	Other Problems	48

Part (1)*

Differential Conservation Equations of Fluid Flow

Introduction:

We start our study in this part by answering the next few questions:

A) What is “Fluid Mechanics”?

- The science of “Fluid Mechanics” is a sub-part of the large field of Applied Mechanics (which include Solid Mechanics, Fluid Mechanics, and Quantum Mechanics).
 - Fluid Mechanics is concerned with the behavior of fluids at rest (i.e., Fluid Static’s) or fluids in motion (i.e., Fluid Dynamic’s).
 - Fluids include liquids, gases, vapors or any mixture of them (i.e., multi-phase flow).
 - Liquids with suspended solids or moving solid particles pneumatically (i.e., by using compressed air) are special cases and may also be studied in Fluid Mechanics.
-

B) Why do we study “Fluid Mechanics”?

- We study Fluid Mechanics in order to do an analysis for an existing open/close system or to do a design for a new system which include fluids as the working medium through it. There are very large numbers of applications of systems having fluids in them.
 - The analysis or design includes a comprehensive understanding of the behavior or performance of any system or machine which uses fluids.
 - This analysis or design includes calculating some/all of the flow properties such as velocity and pressure and also finding the effects and interactions between fluids and their surrounding boundaries (which may be either solid surfaces or other fluids).
-

C) How do we study “Fluid Mechanics”?

- We study Fluid Mechanics theoretically by analytical/computational methods, or by experimental or dimensionless grouping methods, or by combination of these methods. Other methods such as flow visualization are also used with the above methods.
 - In the theoretical analysis method we need to have a comprehensive understanding of all of the different conservation equations governing Fluid Mechanics.
 - These conservation equations are: conservation of mass, conservation of linear and angular momentum, and conservation of energy.
 - The theoretical approach is very difficult and is only limited to some few idealized cases especially if the effect of viscosity is involved in the analysis.
 - Computational fluid mechanics is to solve the conservation equations by a computer at some number of nodes (specific points in the flow field). This is also limited to by the need of huge memory and computation times and the use of many idealization assumptions, modeling techniques, and also many correction factors and constants throughout the computations.
 - Experimental analysis studies do not do any idealization/assumptions and do not need a deep understanding of all of the exact equations of the problems under investigation.
 - Experimental work requires great deal of money/time, its results are limited to few measuring points or practical cases/conditions similar to those tested in the experiments.
- Dimensionless analysis is a must for experimental work to save time/efforts and experiments by focusing on main important factors affecting the problem under consideration. Dimensional analysis makes the results as widely applicable as possible.

* Ref.:(1) Bruce R. Munson, Donald F. Young, Theodore H. Okiishi “Fundamental of Fluid Mechanics” 4th ed., John Wiley & Sons, Inc., 2002.

(2) Frank M. White “Fluid Mechanics”, 4th ed. McGraw Hill, 2002.

D) What is the difference between “Integral Analysis” and “Differential analysis”?

(1) Integral Analysis:

- This approach is very practical and useful in solving many practical fluid problems.
 - Integral analysis uses a finite large fluid control volume (such as the whole section of fluid in a pipe or the whole fluid through the internal parts of a pump or a turbine).
 - We find the integral forms of all the conservation equations governing the fluid flow through this finite control volume (we do not write equations for the solid boundaries).
 - The equations are solved by integration within and along all the surfaces of this control volume. Some average/constant flow properties are usually assumed at the control surfaces which makes doing the integration a simple task.
 - Integral analysis does not require detailed information of the variations of the flow properties (i.e., pressure, velocity, and temperature) within the control volume.
 - The results of integral analysis do not give any exact or detailed equations for the flow properties within every point in the flow field. We find only the average properties or conditions on the surfaces of the finite control volume. We find also the forces and interaction between that control fluid volume and its surroundings.
-

(2) Differential Analysis:

- This approach is used if we need to get all the flow details or general relationships that apply at a point or at a very small region (i.e., infinitesimal volume) in the flow field. Typical examples are pressure and shear stress distributions along the wing of a plane.
 - In differential analysis, we apply the conservation equations to an infinitesimally small control volume or, alternately, to an infinitesimal fluid open/closed system.
 - The resulting equations yield the basic differential equations of fluid motion. These equations are non-linear, partial differential, 2nd order equations. They are to be solved using appropriate known flow boundary conditions at some points in the flow field.
 - The analytical solutions of these differential equations give very detailed information about the flow field at each point. This information is, however, not easily extracted.
 - The differential equations of motion are, however, quite difficult to solve, and very little is known about their general mathematical properties. We can solve them analytically only in some few ideal cases and for very simple geometries.
 - When analytical solution is not possible, the partial differential equations are solved on a computer using the various techniques of Computational Fluid Dynamics (CFD).
-

In this chapter, we will provide an introduction to the differential equations that describe (in detail) the motion of fluids. Differential analysis provides a fundamental basis for the study of fluid mechanics. Unfortunately, we will also find that these equations are rather complicated, partial differential equations that cannot be solved exactly except in a few cases, at least without making some simplifying assumptions. There are some exact solutions for laminar flow that can be obtained, and these have proved to be very useful. In addition by making some simplifying assumptions, many other analytical solutions can be obtained. For example, in some cases, it may be reasonable to assume that the effect of viscosity is small and can be neglected. This rather drastic assumption greatly simplifies the analysis and provides the opportunity to obtain detailed solutions to a variety of complex flow problems. Some examples of these so-called inviscid flow solutions are also described in this chapter.

It is known that for certain types of flows the flow field can be conceptually divided into two regions—a very thin region near the boundaries of the system in which viscous effects are important, and a region away from the boundaries in which the flow is essentially inviscid. By making certain assumptions about the behavior of the fluid in the thin layer near the boundaries, and using the assumption of inviscid flow outside this layer, a large class of problems can be solved using differential analysis. These boundary layer problems are discussed in **Part (4)**. Finally, it is to be noted that with the availability of powerful digital computers it is feasible to attempt to solve the differential equations using the techniques of numerical analysis. Although it is beyond the scope of this book to delve into this approach, which is generally referred to as *computational fluid dynamics* (CFD), the reader should be aware of this approach to complex flow problems. A few additional comments about CFD and other aspects of differential analysis are given in the last section of this chapter.

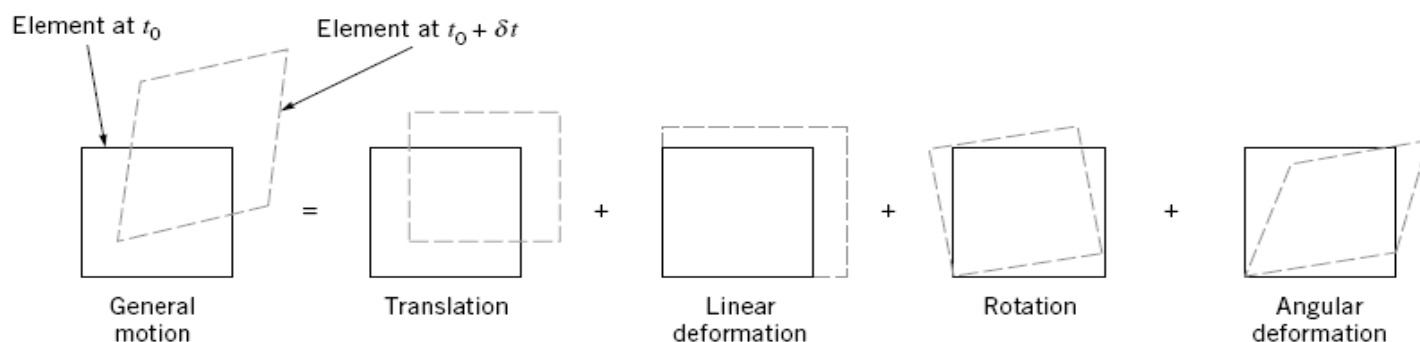
We begin our introduction to differential analysis by reviewing and extending some of the ideas associated with fluid kinematics that were introduced in **2nd year**. With this background the remainder of the chapter will be devoted to the derivation of the basic differential equations (which will be based on the principle of conservation of mass and Newton's second law of motion) and to some applications.

1.1 Fluid Element Kinematics

In this section we will be concerned with the mathematical description of the motion of fluid elements moving in a flow field. A small fluid element in the shape of a cube which is initially in one position will move to another position during a short time interval δt as illustrated in Fig. 1.1. Because of the generally complex velocity variation within the field, we expect the element not only to translate from one position but also to have its volume changed (linear deformation), to rotate, and to undergo a change in shape (angular deformation). Although these movements and deformations occur simultaneously, we can consider each one separately as illustrated in Fig. 1.1. Since element motion and deformation are intimately related to the velocity and variation of velocity throughout the flow field, we will briefly review the manner in which velocity and acceleration fields can be described.

1.1.1 Velocity and Acceleration Fields Revisited

As discussed in detail in **last year**, the velocity field can be described by specifying the velocity \mathbf{V} at all points, and at all times, within the flow field of interest. Thus, in terms of rectangular coordinates, the notation $\mathbf{V}(x, y, z, t)$ means that the velocity of a fluid particle depends on where it is located within the flow field (as determined by its coordinates, x , y , and z) and when it occupies the particular point (as determined by the time, t). As is pointed out **before last year**, this method of describing the fluid motion is called the Eulerian method.



■ **FIGURE 1.1** Types of motion and deformation for a fluid element.

It is also convenient to express the velocity in terms of three rectangular components so that

$$\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}} \quad (1.1)$$

where u , v , and w are the velocity components in the x , y , and z directions, respectively, and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the corresponding unit vectors. Of course, each of these components will, in general, be a function of x , y , z , and t . One of the goals of differential analysis is to determine how these velocity components specifically depend on x , y , z , and t for a particular problem.

With this description of the velocity field it was also shown **before last year** that the acceleration of a fluid particle can be expressed as

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u \frac{\partial \mathbf{V}}{\partial x} + v \frac{\partial \mathbf{V}}{\partial y} + w \frac{\partial \mathbf{V}}{\partial z} \quad (1.2)$$

and in component form:

$$a_x = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \quad (1.3a)$$

$$a_y = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \quad (1.3b)$$

$$a_z = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \quad (1.3c)$$

The acceleration is also concisely expressed as

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt} \quad (1.4)$$

where the operator

$$\frac{D(\)}{Dt} = \frac{\partial(\)}{\partial t} + u \frac{\partial(\)}{\partial x} + v \frac{\partial(\)}{\partial y} + w \frac{\partial(\)}{\partial z} \quad (1.5)$$

is termed the *material derivative*, or *substantial derivative*. In vector notation

$$\frac{D(\)}{Dt} = \frac{\partial(\)}{\partial t} + (\mathbf{V} \cdot \nabla)(\) \quad (1.6)$$

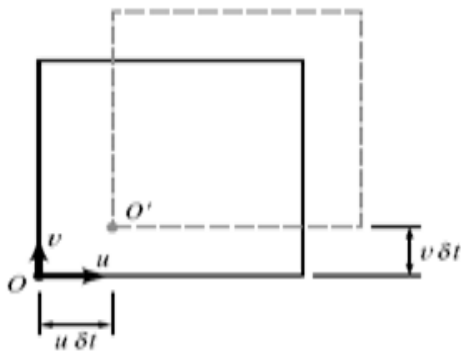
where the gradient operator, $\nabla(\)$, is

$$\nabla(\) = \frac{\partial(\)}{\partial x} \hat{\mathbf{i}} + \frac{\partial(\)}{\partial y} \hat{\mathbf{j}} + \frac{\partial(\)}{\partial z} \hat{\mathbf{k}} \quad (1.7)$$

which was introduced in **2nd year** . . . As we will see in the following sections, the motion and deformation of a fluid element depend on the velocity field. The relationship between the motion and the forces causing the motion depends on the acceleration field.

1.1.2 Linear Motion and Deformation

The simplest type of motion that a fluid element can undergo is translation, as illustrated in Fig. 1.2. In a small time interval δt a particle located at point O will move to point O' as is illustrated in the figure. If all points in the element have the same velocity (which is only true if there are no velocity gradients), then the element will simply translate from one position to another. However, because of the presence of velocity gradients, the element will generally be deformed and rotated as it moves. For example, consider the effect of a single



■ FIGURE 1.2 Translation of a fluid element.

velocity gradient, $\partial u/\partial x$, on a small cube having sides δx , δy , and δz . As is shown in Fig. 1.3a, if the x component of velocity of O and B is u , then at nearby points A and C the x component of the velocity can be expressed as $u + (\partial u/\partial x) \delta x$. This difference in velocity causes a “stretching” of the volume element by an amount $(\partial u/\partial x)(\delta x)(\delta t)$ during the short time interval δt in which line OA stretches to OA' and BC to BC' (Fig. 1.3b). The corresponding change in the original volume, $\delta \Psi = \delta x \delta y \delta z$, would be

$$\text{Change in } \delta \Psi = \left(\frac{\partial u}{\partial x} \delta x \right) (\delta y \delta z) (\delta t)$$

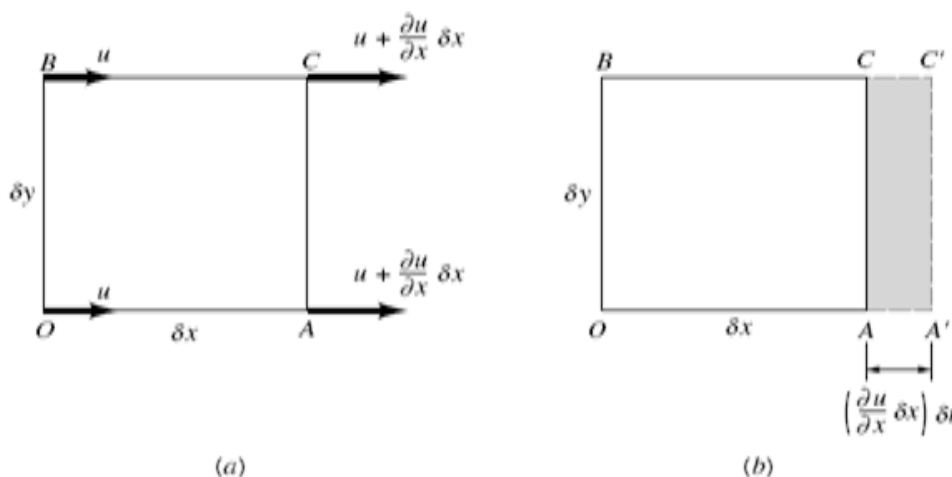
and the *rate* at which the volume $\delta \Psi$ is changing *per unit volume* due to the gradient $\partial u/\partial x$ is

$$\frac{1}{\delta \Psi} \frac{d(\delta \Psi)}{dt} = \lim_{\delta t \rightarrow 0} \left[\frac{(\partial u/\partial x) \delta t}{\delta t} \right] = \frac{\partial u}{\partial x} \quad (1.8)$$

If velocity gradients $\partial/\partial y$ and $\partial w/\partial z$ are also present, then using a similar analysis it follows that, in the general case,

$$\frac{1}{\delta \Psi} \frac{d(\delta \Psi)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{V} \quad (1.9)$$

This rate of change of the volume per unit volume is called the *volumetric dilatation rate*. Thus, we see that the volume of a fluid may change as the element moves from one location to another in the flow field. However, for an *incompressible fluid* the volumetric dilatation rate is zero, since the element volume cannot change without a change in fluid density (the element mass must be conserved). Variations in the velocity in the direction of the velocity, as represented by the derivatives $\partial u/\partial x$, $\partial v/\partial y$, and $\partial w/\partial z$, simply cause a *linear deformation* of the element in the sense that the shape of the element does not change. Cross



■ FIGURE 1.3 Linear deformation of a fluid element.

derivatives, such as $\partial u/\partial y$ and $\partial v/\partial x$, will cause the element to rotate and generally to undergo an *angular deformation*, which changes the shape of the element.

1.1.3 Angular Motion and Deformation

For simplicity we will consider motion in the x - y plane, but the results can be readily extended to the more general case. The velocity variation that causes rotation and angular deformation is illustrated in Fig. 1.4a. In a short time interval δt the line segments OA and OB will rotate through the angles $\delta\alpha$ and $\delta\beta$ to the new positions OA' and OB' , as is shown in Fig. 1.4b. The angular velocity of line OA , ω_{OA} , is

$$\omega_{OA} = \lim_{\delta t \rightarrow 0} \frac{\delta\alpha}{\delta t}$$

For small angles

$$\tan \delta\alpha \approx \delta\alpha = \frac{(\partial v/\partial x) \delta x \delta t}{\delta x} = \frac{\partial v}{\partial x} \delta t \quad (1.10)$$

so that

$$\omega_{OA} = \lim_{\delta t \rightarrow 0} \left[\frac{(\partial v/\partial x) \delta t}{\delta t} \right] = \frac{\partial v}{\partial x}$$

Note that if $\partial v/\partial x$ is positive, ω_{OA} will be counterclockwise. Similarly, the angular velocity of the line OB is

$$\omega_{OB} = \lim_{\delta t \rightarrow 0} \frac{\delta\beta}{\delta t}$$

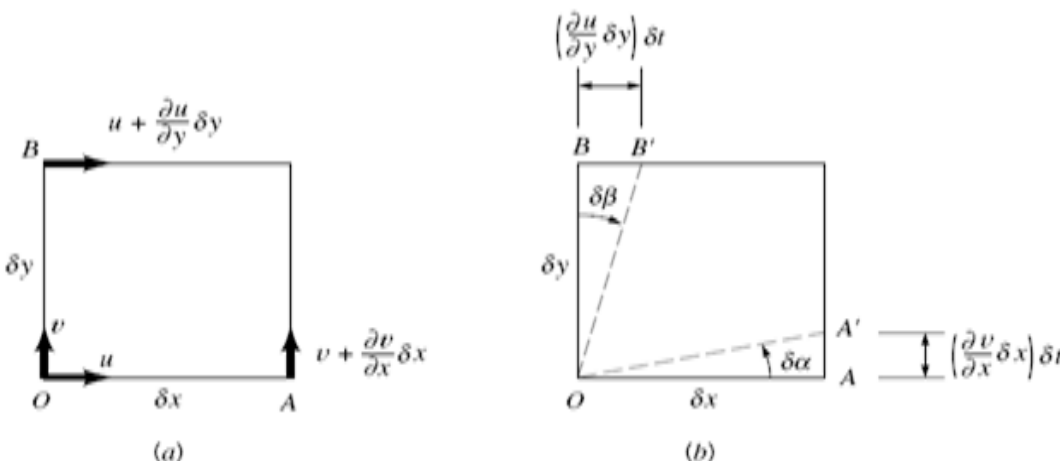
and

$$\tan \delta\beta \approx \delta\beta = \frac{(\partial u/\partial y) \delta y \delta t}{\delta y} = \frac{\partial u}{\partial y} \delta t \quad (1.11)$$

so that

$$\omega_{OB} = \lim_{\delta t \rightarrow 0} \left[\frac{(\partial u/\partial y) \delta t}{\delta t} \right] = \frac{\partial u}{\partial y}$$

In this instance if $\partial u/\partial y$ is positive, ω_{OB} will be clockwise. The *rotation*, ω_z , of the element about the z axis is defined as the average of the angular velocities ω_{OA} and ω_{OB} of the two



■ FIGURE 1.4
Angular motion and
deformation of a
fluid element.

mutually perpendicular lines OA and OB .¹ Thus, if counterclockwise rotation is considered to be positive, it follows that

$$\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \quad (1.12)$$

Rotation of the field element about the other two coordinate axes can be obtained in a similar manner with the result that for rotation about the x axis

$$\omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \quad (1.13)$$

and for rotation about the y axis

$$\omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \quad (1.14)$$

The three components, ω_x , ω_y , and ω_z can be combined to give the rotation vector, $\boldsymbol{\omega}$, in the form

$$\boldsymbol{\omega} = \omega_x \hat{\mathbf{i}} + \omega_y \hat{\mathbf{j}} + \omega_z \hat{\mathbf{k}} \quad (1.15)$$

An examination of this result reveals that $\boldsymbol{\omega}$ is equal to one-half the curl of the velocity vector. That is,

$$\boldsymbol{\omega} = \frac{1}{2} \text{curl } \mathbf{V} = \frac{1}{2} \nabla \times \mathbf{V} \quad (1.16)$$

since by definition of the vector operator $\nabla \times \mathbf{V}$

$$\begin{aligned} \frac{1}{2} \nabla \times \mathbf{V} &= \frac{1}{2} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\ &= \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{\mathbf{i}} + \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{\mathbf{j}} + \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{\mathbf{k}} \end{aligned}$$

The *vorticity*, $\boldsymbol{\zeta}$, is defined as a vector that is twice the rotation vector; that is,

$$\boldsymbol{\zeta} = 2 \boldsymbol{\omega} = \nabla \times \mathbf{V} \quad (1.17)$$

The use of the vorticity to describe the rotational characteristics of the fluid simply eliminates the $(\frac{1}{2})$ factor associated with the rotation vector.

We observe from Eq. 1.12 that the fluid element will rotate about the z axis as an *undeformed* block (i.e., $\omega_{OA} = -\omega_{OB}$) only when $\partial u/\partial y = -\partial v/\partial x$. Otherwise the rotation will be associated with an angular deformation. We also note from Eq. 6.12 that when $\partial u/\partial y = \partial v/\partial x$ the rotation around the z axis is zero. More generally if $\nabla \times \mathbf{V} = 0$, then the rotation (and the vorticity) are zero, and flow fields for which this condition applies are termed *irrotational*. We will find in Section 1.4 that the condition of irrotationality often greatly simplifies the analysis of complex flow fields. However, it is probably not immediately obvious why some flow fields would be irrotational, and we will need to examine this concept more fully in Section 1.4.

¹With this definition ω_z can also be interpreted to be the angular velocity of the bisector of the angle between the lines OA and OB .

Example 1.1:

For a certain two-dimensional flow field the velocity is given by the equation

$$\mathbf{V} = 4xy \hat{\mathbf{i}} + 2(x^2 - y^2) \hat{\mathbf{j}}$$

Is this flow irrotational?

SOLUTION

For an irrotational flow the rotation vector, $\boldsymbol{\omega}$, having the components given by Eqs. 1.12, 1.13, and 1.14 must be zero. For the prescribed velocity field

$$u = 4xy \quad v = 2(x^2 - y^2) \quad w = 0$$

and therefore

$$\omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = 0$$

$$\omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) = 0$$

$$\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{1}{2} (4x - 4x) = 0$$

Thus, the flow is irrotational.

(Ans)

It is to be noted that for a two-dimensional flow field (where the flow is in the x - y plane) ω_x and ω_y will always be zero, since by definition of two-dimensional flow u and v are not functions of z , and w is zero. In this instance the condition for irrotationality simply becomes $\omega_z = 0$ or $\partial v / \partial x = \partial u / \partial y$. (Lines OA and OB of Fig. 1.4 rotate with the same speed but in opposite directions so that there is no rotation of the fluid element.)

In addition to the rotation associated with the derivatives $\partial u / \partial y$ and $\partial v / \partial x$, it is observed from Fig. 1.4b that these derivatives can cause the fluid element to undergo an *angular deformation*, which results in a change in shape of the element. The change in the original right angle formed by the lines OA and OB is termed the shearing strain, $\delta\gamma$, and from Fig. 1.4b

$$\delta\gamma = \delta\alpha + \delta\beta$$

where $\delta\gamma$ is considered to be positive if the original right angle is decreasing. The rate of change of $\delta\gamma$ is called the *rate of shearing strain* or the *rate of angular deformation* and is commonly denoted with the symbol $\dot{\gamma}$. The angles $\delta\alpha$ and $\delta\beta$ are related to the velocity gradients through Eqs. 1.10 and 1.11 so that

$$\dot{\gamma} = \lim_{\delta t \rightarrow 0} \frac{\delta\gamma}{\delta t} = \lim_{\delta t \rightarrow 0} \left[\frac{(\partial v / \partial x) \delta t + (\partial u / \partial y) \delta t}{\delta t} \right]$$

and, therefore,

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad (1.18)$$

As we will learn in Section 1.8, the rate of angular deformation is related to a corresponding shearing stress which causes the fluid element to change in shape. From Eq. 1.18 we

note that if $\partial u/\partial y = -\partial v/\partial x$, the rate of angular deformation is zero, and this condition corresponds to the case in which the element is simply rotating as an undeformed block (Eq. 1.12). In the remainder of this chapter we will see how the various kinematical relationships developed in this section play an important role in the development and subsequent analysis of the differential equations that govern fluid motion.

1.2 Conservation of Mass

As is discussed in 2nd year , conservation of mass requires that the mass, M , of a system remain constant as the system moves through the flow field. In equation form this principle is expressed as

$$\frac{DM_{\text{sys}}}{Dt} = 0$$

We found it convenient to use the control volume approach for fluid flow problems, with the control volume representation of the conservation of mass written as

$$\frac{\partial}{\partial t} \int_{\text{cv}} \rho dV + \int_{\text{cs}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA = 0 \quad (1.19)$$

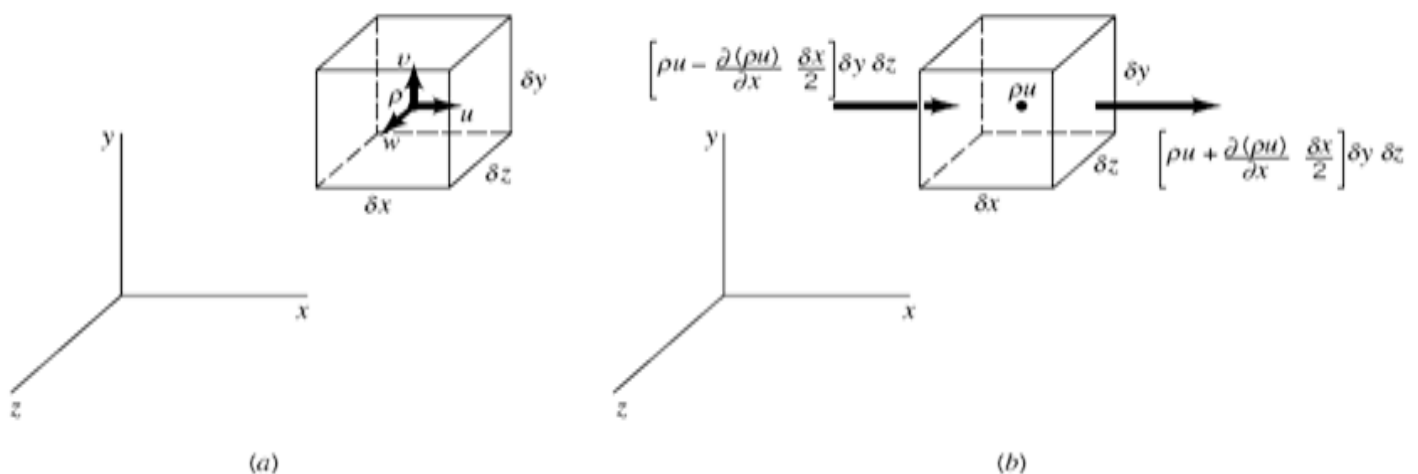
where the equation (commonly called the continuity equation) can be applied to a finite control volume (cv), which is bounded by a control surface (cs). The first integral on the left side of Eq. 1.19 represents the rate at which the mass within the control volume is changing, and the second integral represents the net rate at which mass is flowing out through the control surface (rate of mass outflow – rate of mass inflow). To obtain the differential form of the continuity equation, Eq. 1.19 is applied to an infinitesimal control volume.

1.2.1 Differential Form of Continuity Equation

We will take as our control volume the small, stationary cubical element shown in Fig. 1.5a. At the center of the element the fluid density is ρ and the velocity has components u , v , and w . Since the element is small, the volume integral in Eq. 1.19 can be expressed as

$$\frac{\partial}{\partial t} \int_{\text{cv}} \rho dV \approx \frac{\partial \rho}{\partial t} \delta x \delta y \delta z \quad (1.20)$$

The rate of mass flow through the surfaces of the element can be obtained by considering the flow in each of the coordinate directions separately. For example, in Fig. 1.5b flow in



■ FIGURE 1.5 A differential element for the development of conservation of mass equation.

the x direction is depicted. If we let ρu represent the x component of the mass rate of flow per unit area at the center of the element, then on the right face

$$\rho u|_{x+(\delta x/2)} = \rho u + \frac{\partial(\rho u)}{\partial x} \frac{\delta x}{2} \quad (1.21)$$

and on the left face

$$\rho u|_{x-(\delta x/2)} = \rho u - \frac{\partial(\rho u)}{\partial x} \frac{\delta x}{2} \quad (1.22)$$

Note that we are really using a Taylor series expansion of ρu and neglecting higher order terms such as $(\delta x)^2$, $(\delta x)^3$, and so on. When the right-hand sides of Eqs. 1.21 and 1.22 are multiplied by the area $\delta y \delta z$, the rate at which mass is crossing the right and left sides of the element are obtained as is illustrated in Fig. 1.5b. When these two expressions are combined, the net rate of mass flowing from the element through the two surfaces can be expressed as

$$\begin{aligned} \text{Net rate of mass} \\ \text{outflow in } x \text{ direction} &= \left[\rho u + \frac{\partial(\rho u)}{\partial x} \frac{\delta x}{2} \right] \delta y \delta z \\ &\quad - \left[\rho u - \frac{\partial(\rho u)}{\partial x} \frac{\delta x}{2} \right] \delta y \delta z = \frac{\partial(\rho u)}{\partial x} \delta x \delta y \delta z \quad (1.23) \end{aligned}$$

For simplicity, only flow in the x direction has been considered in Fig. 1.5b, but, in general, there will also be flow in the y and z directions. An analysis similar to the one used for flow in the x direction shows that

$$\begin{aligned} \text{Net rate of mass} \\ \text{outflow in } y \text{ direction} &= \frac{\partial(\rho v)}{\partial y} \delta x \delta y \delta z \quad (1.24) \end{aligned}$$

and

$$\begin{aligned} \text{Net rate of mass} \\ \text{outflow in } z \text{ direction} &= \frac{\partial(\rho w)}{\partial z} \delta x \delta y \delta z \quad (1.25) \end{aligned}$$

Thus,

$$\begin{aligned} \text{Net rate of} \\ \text{mass outflow} &= \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] \delta x \delta y \delta z \quad (1.26) \end{aligned}$$

From Eqs. 1.19, 1.20, and 1.26 it now follows that the differential equation for conservation of mass is

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0} \quad (1.27)$$

As previously mentioned, this equation is also commonly referred to as the continuity equation.

The continuity equation is one of the fundamental equations of fluid mechanics and, as expressed in Eq. 1.27, is valid for steady or unsteady flow, and compressible or incompressible fluids. In vector notation, Eq. 1.27 can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0 \quad (1.28)$$

Two special cases are of particular interest. For *steady* flow of *compressible* fluids

$$\nabla \cdot \rho \mathbf{V} = 0$$

or

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad (1.29)$$

This follows since by definition ρ is not a function of time for steady flow, but could be a function of position. For *incompressible* fluids the fluid density, ρ , is a constant throughout the flow field so that Eq. 1.28 becomes

$$\nabla \cdot \mathbf{V} = 0 \quad (1.30)$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (1.31)$$

Equation 1.31 applies to both steady and unsteady flow of incompressible fluids. Note that Eq. 1.31 is the same as that obtained by setting the volumetric dilatation rate (Eq. 1.9) equal to zero. This result should not be surprising since both relationships are based on conservation of mass for incompressible fluids. However, the expression for the volumetric dilatation rate was developed from a system approach, whereas Eq. 1.31 was developed from a control volume approach. In the former case the deformation of a particular differential mass of fluid was studied, and in the latter case mass flow through a fixed differential volume was studied.

Example 1.2a:

Given the eulerian velocity-vector field

$$\mathbf{V} = 3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}$$

find the acceleration of a particle.

Solution

First note the specific given components

$$u = 3t \quad v = xz \quad w = ty^2$$

Then evaluate the vector derivatives required

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{i} \frac{\partial u}{\partial t} + \mathbf{j} \frac{\partial v}{\partial t} + \mathbf{k} \frac{\partial w}{\partial t} = 3\mathbf{i} + y^2\mathbf{k}$$

$$\frac{\partial \mathbf{V}}{\partial x} = z\mathbf{j} \quad \frac{\partial \mathbf{V}}{\partial y} = 2ty\mathbf{k} \quad \frac{\partial \mathbf{V}}{\partial z} = x\mathbf{j}$$

This could have been worse: There are only five terms in all, whereas there could have been as many as twelve. Substitute directly

$$\frac{d\mathbf{V}}{dt} = (3\mathbf{i} + y^2\mathbf{k}) + (3t)(z\mathbf{j}) + (xz)(2ty\mathbf{k}) + (ty^2)(x\mathbf{j})$$

Collect terms for the final result

$$\frac{d\mathbf{V}}{dt} = 3\mathbf{i} + (3tz + txy^2)\mathbf{j} + (2xyzt + y^2)\mathbf{k} \quad \text{Ans.}$$

Assuming that \mathbf{V} is valid everywhere as given, this acceleration applies to all positions and times within the flow field.

Example 1.2b:

Under what conditions does the velocity field

$$\mathbf{V} = (a_1x + b_1y + c_1z)\mathbf{i} + (a_2x + b_2y + c_2z)\mathbf{j} + (a_3x + b_3y + c_3z)\mathbf{k}$$

where $a_1, b_1, \text{ etc.} = \text{const}$, represent an incompressible flow which conserves mass?

Solution

Recalling that $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, we see that $u = (a_1x + b_1y + c_1z)$, etc. Substituting into Eq. for incompressible continuity, we obtain

$$\frac{\partial}{\partial x} (a_1x + b_1y + c_1z) + \frac{\partial}{\partial y} (a_2x + b_2y + c_2z) + \frac{\partial}{\partial z} (a_3x + b_3y + c_3z) = 0$$

or $a_1 + b_2 + c_3 = 0$ Ans.

At least two of constants a_1, b_2 , and c_3 must have opposite signs. Continuity imposes no restrictions whatever on constants b_1, c_1, a_2, c_2, a_3 , and b_3 , which do not contribute to a mass increase or decrease of a differential element.

Example 1.2c:

An incompressible velocity field is given by

$$u = a(x^2 - y^2) \quad v \text{ unknown} \quad w = b$$

where a and b are constants. What must the form of the velocity component v be?

Solution

Again Eq. (1.31) applies

$$\frac{\partial}{\partial x} (ax^2 - ay^2) + \frac{\partial v}{\partial y} + \frac{\partial b}{\partial z} = 0$$

or
$$\frac{\partial v}{\partial y} = -2ax \tag{1}$$

This is easily integrated partially with respect to y

$$v(x, y, z, t) = -2axy + f(x, z, t) \tag{Ans.}$$

†This is a very realistic flow which simulates the turning of an inviscid fluid through a 60° angle

This is the only possible form for v which satisfies the incompressible continuity equation. The function of integration f is entirely arbitrary since it vanishes when v is differentiated with respect to y .†

Example 1.2d:

A centrifugal impeller of 40-cm diameter is used to pump hydrogen at 15°C and 1-atm pressure. What is the maximum allowable impeller rotational speed to avoid compressibility effects at the blade tips?

Solution

The speed of sound of hydrogen for these conditions is $a = 1300$ m/s. Assume that the gas velocity leaving the impeller is approximately equal to the impeller-tip speed

$$V = \Omega r = \frac{1}{2}\Omega D$$

Our rule of thumb, $M\# < 0.3$, neglects compressibility if

$$V = \frac{1}{2}\Omega D \leq 0.3a = 390 \text{ m/s}$$

or
$$\frac{1}{2}\Omega(0.4 \text{ m}) \leq 390 \text{ m/s} \quad \Omega \leq 1950 \text{ rad/s}$$

Thus we estimate the allowable speed to be quite large

$$\Omega \leq 310 \text{ r/s (18,600 r/min)} \tag{Ans.}$$

An impeller moving at this speed in air would create shock waves at the tips but not in a light gas like hydrogen.

Example 1.2e:

The velocity components for a certain incompressible, steady flow field are

$$u = x^2 + y^2 + z^2$$

$$v = xy + yz + z$$

$$w = ?$$

Determine the form of the z component, w , required to satisfy the continuity equation.

SOLUTION

Any physically possible velocity distribution must for an incompressible fluid satisfy conservation of mass as expressed by the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

For the given velocity distribution

$$\frac{\partial u}{\partial x} = 2x \quad \text{and} \quad \frac{\partial v}{\partial y} = x + z$$

so that the required expression for $\partial w/\partial z$ is

$$\frac{\partial w}{\partial z} = -2x - (x + z) = -3x - z$$

Integration with respect to z yields

$$w = -3xz - \frac{z^2}{2} + f(x, y) \quad (\text{Ans})$$

The third velocity component cannot be explicitly determined since the function $f(x, y)$ can have any form and conservation of mass will still be satisfied. The specific form of this function will be governed by the flow field described by these velocity components—that is, some additional information is needed to completely determine w .

1.2.2 Cylindrical Polar Coordinates

For some problems it is more convenient to express the various differential relationships in cylindrical polar coordinates rather than Cartesian coordinates. As is shown in Fig. 1.6, with cylindrical coordinates a point is located by specifying the coordinates r , θ , and z . The coordinate r is the radial distance from the z axis, θ is the angle measured from a line parallel to the x axis (with counterclockwise taken as positive), and z is the coordinate along the z axis. The velocity components, as sketched in Fig. 1.6, are the radial velocity, v_r , the tangential velocity, v_θ , and the axial velocity, v_z . Thus, the velocity at some arbitrary point P can be expressed as

$$\mathbf{V} = v_r \hat{\mathbf{e}}_r + v_\theta \hat{\mathbf{e}}_\theta + v_z \hat{\mathbf{e}}_z \quad (1.32)$$

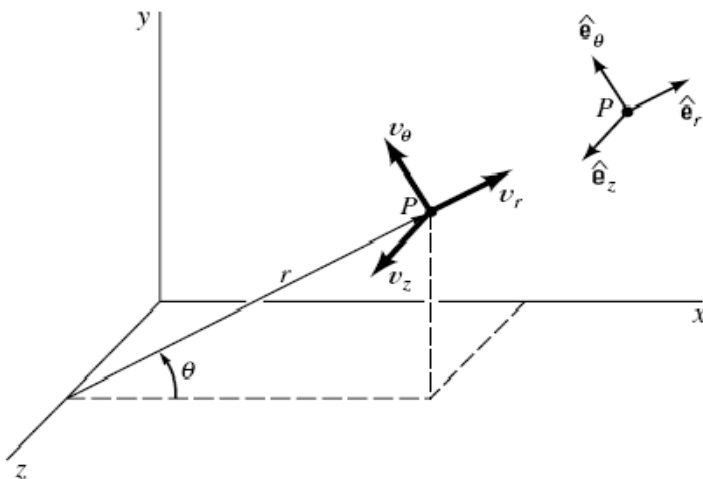
where $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, and $\hat{\mathbf{e}}_z$ are the unit vectors in the r , θ , and z directions, respectively, as are illustrated in Fig. 1.6. The use of cylindrical coordinates is particularly convenient when the boundaries of the flow system are cylindrical. Several examples illustrating the use of cylindrical coordinates will be given in succeeding sections in this chapter.

The differential form of the continuity equation in cylindrical coordinates is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial (r\rho v_r)}{\partial r} + \frac{1}{r} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0 \quad (1.33)$$

This equation can be derived by following the same procedure used in the preceding section (see **Problem 1.17**). For steady, compressible flow

$$\frac{1}{r} \frac{\partial (r\rho v_r)}{\partial r} + \frac{1}{r} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0 \quad (1.34)$$



■ **FIGURE 1.6** The representation of velocity components in cylindrical polar coordinates:

For incompressible fluids (for steady or unsteady flow)

$$\frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \quad (1.35)$$

1.2.3 The Stream Function

Steady, incompressible, plane, two-dimensional flow represents one of the simplest types of flow of practical importance. By plane, two-dimensional flow we mean that there are only two velocity components, such as u and v , when the flow is considered to be in the x - y plane. For this flow the continuity equation, Eq. 1.31, reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1.36)$$

We still have two variables, u and v , to deal with, but they must be related in a special way as indicated by Eq. 1.36. This equation suggests that if we define a function $\psi(x, y)$, called the *stream function*, which relates the velocities as

$$\boxed{u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}} \quad (1.37)$$

then the continuity equation is identically satisfied. This conclusion can be verified by simply substituting the expressions for u and v into Eq. 1.36 so that

$$\frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial \psi}{\partial x} \right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

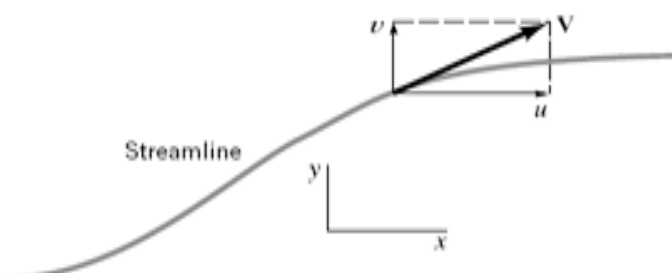
Thus, whenever the velocity components are defined in terms of the stream function we know that conservation of mass will be satisfied. Of course, we still do not know what $\psi(x, y)$ is for a particular problem, but at least we have simplified the analysis by having to determine only one unknown function, $\psi(x, y)$, rather than the two functions, $u(x, y)$ and $v(x, y)$.

Another particular advantage of using the stream function is related to the fact that *lines along which ψ is constant are streamlines*. Recall from **second year** that streamlines are lines in the flow field that are everywhere tangent to the velocities, as is illustrated in Fig. 1.7. It follows from the definition of the streamline that the slope at any point along a streamline is given by

$$\frac{dy}{dx} = \frac{v}{u}$$

The change in the value of ψ as we move from one point (x, y) to a nearby point $(x + dx, y + dy)$ is given by the relationship:

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v dx + u dy$$



■ FIGURE 1.7 Velocity and velocity components along a streamline.

Along a line of constant ψ we have $d\psi = 0$ so that

$$-v dx + u dy = 0$$

and, therefore, along a line of constant ψ

$$\frac{dy}{dx} = \frac{v}{u}$$

which is the defining equation for a streamline. Thus, if we know the function $\psi(x, y)$ we can plot lines of constant ψ to provide the family of streamlines that are helpful in visualizing the pattern of flow. There are an infinite number of streamlines that make up a particular flow field, since for each constant value assigned to ψ a streamline can be drawn.

The actual numerical value associated with a particular streamline is not of particular significance, but the change in the value of ψ is related to the volume rate of flow. Consider two closely spaced streamlines, shown in Fig. 1.8a. The lower streamline is designated ψ and the upper one $\psi + d\psi$. Let dq represent the volume rate of flow (per unit width perpendicular to the x - y plane) passing between the two streamlines. Note that flow never crosses streamlines, since by definition the velocity is tangent to the streamline. From conservation of mass we know that the inflow, dq , crossing the arbitrary surface AC of Fig. 1.8a must equal the net outflow through surfaces AB and BC . Thus,

$$dq = u dy - v dx$$

or in terms of the stream function

$$dq = \frac{\partial\psi}{\partial y} dy + \frac{\partial\psi}{\partial x} dx \quad (1.38)$$

The right-hand side of Eq. 1.38 is equal to $d\psi$ so that

$$dq = d\psi \quad (1.39)$$

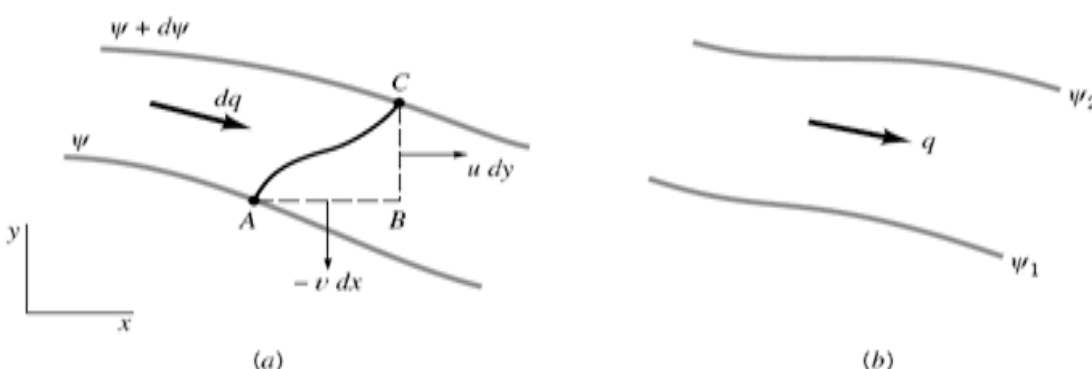
Thus, the volume rate of flow, q , between two streamlines such as ψ_1 and ψ_2 of Fig. 1.8b can be determined by integrating Eq. 1.39 to yield

$$q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1 \quad (1.40)$$

If the upper streamline, ψ_2 , has a value greater than the lower streamline, ψ_1 , then q is positive, which indicates that the flow is from left to right. For $\psi_1 > \psi_2$ the flow is from right to left.

In cylindrical coordinates the continuity equation (Eq. 1.35) for incompressible, plane, two-dimensional flow reduces to

$$\frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = 0 \quad (1.41)$$



■ FIGURE 1.8
The flow between
two streamlines.

and the velocity components, v_r and v_θ , can be related to the stream function, $\psi(r, \theta)$, through the equations

$$\boxed{v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad v_\theta = -\frac{\partial \psi}{\partial r}} \quad (1.42)$$

Substitution of these expressions for the velocity components into Eq. 1.41 shows that the continuity equation is identically satisfied. The stream function concept can be extended to axisymmetric flows, such as flow in pipes or flow around bodies of revolution, and to two-dimensional compressible flows. However, the concept is not applicable to general three-dimensional flows.

Example 1.3:

The velocity components in a steady, incompressible, two-dimensional flow field are

$$u = 2y$$

$$v = 4x$$

Determine the corresponding stream function and show on a sketch several streamlines. Indicate the direction of flow along the streamlines.

SOLUTION

From the definition of the stream function (Eqs. 1.37)

$$u = \frac{\partial \psi}{\partial y} = 2y$$

and

$$v = -\frac{\partial \psi}{\partial x} = 4x$$

The first of these equations can be integrated to give

$$\psi = y^2 + f_1(x)$$

where $f_1(x)$ is an arbitrary function of x . Similarly from the second equation

$$\psi = -2x^2 + f_2(y)$$

where $f_2(y)$ is an arbitrary function of y . It now follows that in order to satisfy both expressions for the stream function

$$\psi = -2x^2 + y^2 + C \quad (\text{Ans})$$

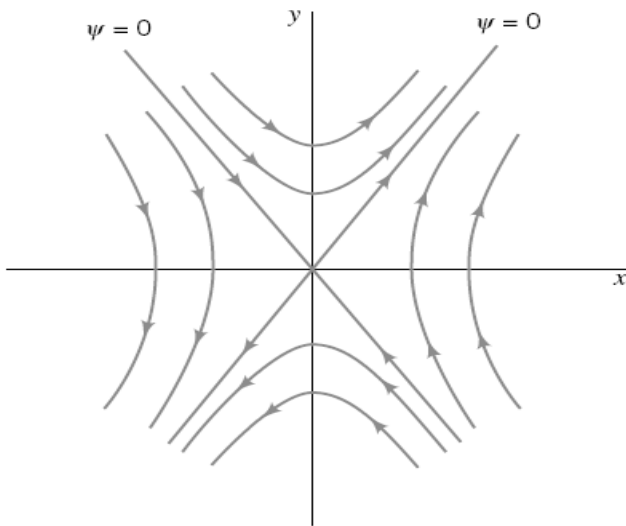
where C is an arbitrary constant.

Since the velocities are related to the derivatives of the stream function, an arbitrary constant can always be added to the function, and the value of the constant is actually of no consequence. Usually, for simplicity, we set $C = 0$ so that for this particular example the simplest form for the stream function is

$$\psi = -2x^2 + y^2 \quad (\mathbf{1}) \quad (\text{Ans})$$

Either answer indicated would be acceptable.

Streamlines can now be determined by setting $\psi = \text{constant}$ and plotting the resulting curve. With the above expression for ψ (with $C = 0$) the value of ψ at the origin is zero so



E 1.3

■ FIGURE E 6.3

that the equation of the streamline passing through the origin (the $\psi = 0$ streamline) is

$$0 = -2x^2 + y^2$$

or

$$y = \pm \sqrt{2}x$$

Other streamlines can be obtained by setting ψ equal to various constants. It follows from Eq. 1 that the equations of these streamlines (for $\psi \neq 0$) can be expressed in the form

$$\frac{y^2}{\psi} - \frac{x^2}{\psi/2} = 1$$

which we recognize as the equation of a hyperbola. Thus, the streamlines are a family of hyperbolas with the $\psi = 0$ streamlines as asymptotes. Several of the streamlines are plotted in Fig. E1.3. Since the velocities can be calculated at any point, the direction of flow along a given streamline can be easily deduced. For example, $v = -\partial\psi/\partial x = 4x$ so that $v > 0$ if $x > 0$ and $v < 0$ if $x < 0$. The direction of flow is indicated on the figure.

1.3 Conservation of Linear Momentum

To develop the differential, linear momentum equations we can start with the linear momentum equation

$$\mathbf{F} = \frac{D\mathbf{P}}{Dt} \Big|_{\text{sys}} \quad (1.43)$$

where \mathbf{F} is the resultant force acting on a fluid mass, \mathbf{P} is the linear momentum defined as

$$\mathbf{P} = \int_{\text{sys}} \mathbf{V} \, dm$$

and the operator $D(\)/Dt$ is the material derivative (see Section 1.2.1). In the last chapter it was demonstrated how Eq. 1.43 in the form

$$\sum \mathbf{F}_{\text{contents of the control volume}} = \frac{\partial}{\partial t} \int_{\text{cv}} \mathbf{V} \rho \, dV + \int_{\text{cs}} \mathbf{V} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA \quad (1.44)$$

could be applied to a finite control volume to solve a variety of flow problems. To obtain the differential form of the linear momentum equation, we can either apply Eq 1.43 to a differential system, consisting of a mass, δm , or apply Eq. 1.44 to an infinitesimal control volume, δV , which initially bounds the mass δm . It is probably simpler to use the system approach since application of Eq. 1.43 to the differential mass, δm , yields

$$\delta \mathbf{F} = \frac{D(\mathbf{V} \delta m)}{Dt}$$

where $\delta \mathbf{F}$ is the resultant force acting on δm . Using this system approach δm can be treated as a constant so that

$$\delta \mathbf{F} = \delta m \frac{D\mathbf{V}}{Dt}$$

But $D\mathbf{V}/Dt$ is the acceleration, \mathbf{a} , of the element. Thus,

$$\delta \mathbf{F} = \delta m \mathbf{a} \quad (1.45)$$

which is simply Newton's second law applied to the mass δm . This is the same result that would be obtained by applying Eq. 1.44 to an infinitesimal control volume (see Ref. 1). Before we can proceed, it is necessary to examine how the force $\delta \mathbf{F}$ can be most conveniently expressed.

1.3.1 Description of Forces Acting on the Differential Element

In general, two types of forces need to be considered: *surface forces*, which act on the surface of the differential element, and *body forces*, which are distributed throughout the element. For our purpose, the only body force, $\delta \mathbf{F}_b$, of interest is the weight of the element, which can be expressed as

$$\delta \mathbf{F}_b = \delta m \mathbf{g} \quad (1.46)$$

where \mathbf{g} is the vector representation of the acceleration of gravity. In component form

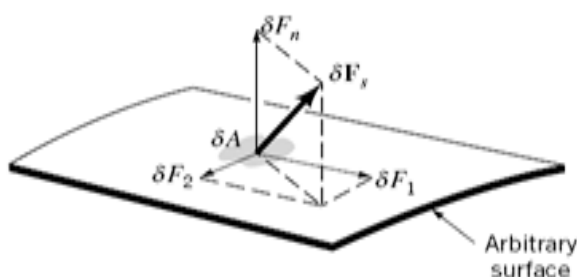
$$\delta F_{bx} = \delta m g_x \quad (1.47a)$$

$$\delta F_{by} = \delta m g_y \quad (1.47b)$$

$$\delta F_{bz} = \delta m g_z \quad (1.47c)$$

where g_x , g_y , and g_z are the components of the acceleration of gravity vector in the x , y , and z directions, respectively.

Surface forces act on the element as a result of its interaction with its surroundings. At any arbitrary location within a fluid mass, the force acting on a small area, δA , which lies in an arbitrary surface, can be represented by $\delta \mathbf{F}_s$, as is shown in Fig. 1.9. In general, $\delta \mathbf{F}_s$ will be inclined with respect to the surface. The force $\delta \mathbf{F}_s$ can be resolved into three components,



■ FIGURE 1.9 Component of force acting on an arbitrary differential area.

δF_n , δF_1 , and δF_2 , where δF_n is normal to the area, δA , and δF_1 and δF_2 are parallel to the area and orthogonal to each other. The *normal stress*, σ_n , is defined as

$$\sigma_n = \lim_{\delta A \rightarrow 0} \frac{\delta F_n}{\delta A}$$

and the *shearing stresses* are defined as

$$\tau_1 = \lim_{\delta A \rightarrow 0} \frac{\delta F_1}{\delta A}$$

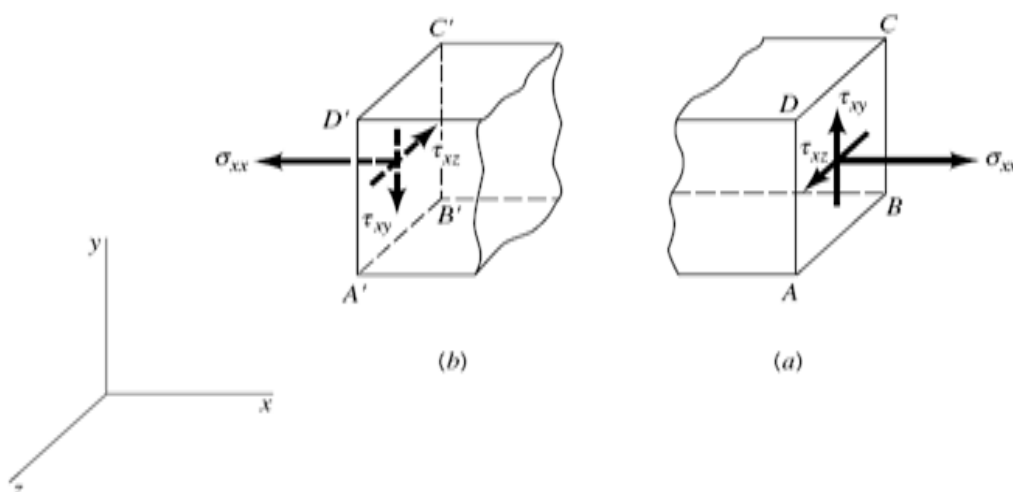
and

$$\tau_2 = \lim_{\delta A \rightarrow 0} \frac{\delta F_2}{\delta A}$$

We will use σ for normal stresses and τ for shearing stresses. The intensity of the force per unit area at a point in a body can thus be characterized by a normal stress and two shearing stresses, if the orientation of the area is specified. For purposes of analysis it is usually convenient to reference the area to the coordinate system. For example, for the rectangular coordinate system shown in Fig. 1.10 we choose to consider the stresses acting on planes parallel to the coordinate planes. On the plane $ABCD$ of Fig. 1.10a, which is parallel to the $y-z$ plane, the normal stress is denoted σ_{xx} and the shearing stresses are denoted as τ_{xy} and τ_{xz} . To easily identify the particular stress component we use a double subscript notation. The first subscript indicates the direction of the *normal* to the plane on which the stress acts, and the second subscript indicates the direction of the stress. Thus, normal stresses have repeated subscripts, whereas the subscripts for the shearing stresses are always different.

It is also necessary to establish a sign convention for the stresses. We define the positive direction for the stress as the positive coordinate direction on the surfaces for which the outward normal is in the positive coordinate direction. This is the case illustrated in Fig. 1.10a where the outward normal to the area $ABCD$ is in the positive x direction. The positive directions for σ_{xx} , τ_{xy} , and τ_{xz} are as shown in Fig. 1.10a. If the outward normal points in the negative coordinate direction, as in Fig. 1.10b for the area $A'B'C'D'$, then the stresses are considered positive if directed in the negative coordinate directions. Thus, the stresses shown in Fig. 1.10b are considered to be positive when directed as shown. Note that positive normal stresses are tensile stresses; that is, they tend to “stretch” the material.

It should be emphasized that the state of stress at a point in a material is not completely defined by simply three components of a “stress vector.” This follows, since any particular stress vector depends on the orientation of the plane passing through the point. However, it can be shown that the normal and shearing stresses acting on *any* plane passing through a



■ FIGURE 1.10
Double subscript notation for stresses.

point can be expressed in terms of the stresses acting on three orthogonal planes passing through the point (Ref. 2).

We now can express the surface forces acting on a small cubical element of fluid in terms of the stresses acting on the faces of the element as shown in Fig. 1.11. It is expected that in general the stresses will vary from point to point within the flow field. Thus, we will express the stresses on the various faces in terms of the corresponding stresses at the center of the element of Fig. 1.11 and their gradients in the coordinate directions. For simplicity only the forces in the x direction are shown. Note that the stresses must be multiplied by the area on which they act to obtain the force. Summing all these forces in the x direction yields

$$\delta F_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \delta x \delta y \delta z \quad (1.48a)$$

for the resultant surface force in the x direction. In a similar manner the resultant surface forces in the y and z directions can be obtained and expressed as

$$\delta F_{sy} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) \delta x \delta y \delta z \quad (1.48b)$$

and

$$\delta F_{sz} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \delta x \delta y \delta z \quad (1.48c)$$

The resultant surface force can now be expressed as

$$\delta \mathbf{F}_s = \delta F_{sx} \hat{\mathbf{i}} + \delta F_{sy} \hat{\mathbf{j}} + \delta F_{sz} \hat{\mathbf{k}} \quad (1.49)$$

and this force combined with the body force, $\delta \mathbf{F}_b$, yields the resultant force, $\delta \mathbf{F}$, acting on the differential mass, δm . That is, $\delta \mathbf{F} = \delta \mathbf{F}_s + \delta \mathbf{F}_b$.

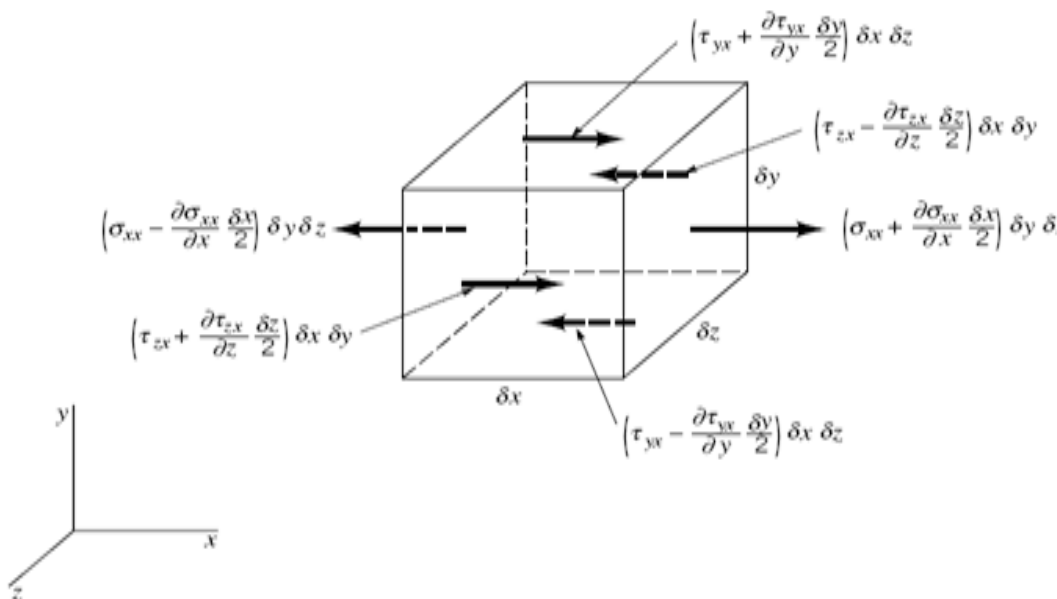
1.3.2 Equations of Motion

The expressions for the body and surface forces can now be used in conjunction with Eq. 1.45 to develop the equations of motion. In component form Eq. 1.45 can be written as

$$\delta F_x = \delta m a_x$$

$$\delta F_y = \delta m a_y$$

$$\delta F_z = \delta m a_z$$



■ FIGURE 1.11
Surface forces in the x direction acting on a fluid element.

where $\delta m = \rho \delta x \delta y \delta z$, and the acceleration components are given by Eq. 1.3. It now follows (using Eqs. 1.47 and 1.48 for the forces on the element) that

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \quad (1.50a)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \quad (1.50b)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) \quad (1.50c)$$

where the element volume $\delta x \delta y \delta z$ cancels out.

Equations 1.50 are the general differential equations of motion for a fluid. In fact, they are applicable to any continuum (solid or fluid) in motion or at rest. However, before we can use the equations to solve specific problems, some additional information about the stresses must be obtained. Otherwise, we will have more unknowns (all of the stresses and velocities and the density) than equations. It should not be too surprising that the differential analysis of fluid motion is complicated. We are attempting to describe, in detail, complex fluid motion.

Note (1): The Differential continuity equation is the same for both viscous or inviscid flow (why?).

Note (2): Up to this point, we did not define the shear stress tensor σ_{ij} in the above linear momentum equations (1.50).

We can have two cases:

Case one:

We define (as in Part (3) of this course) the tensor $\sigma_{ij} = \text{zero}$ for inviscid or ideal or frictionless flows for which there is no effect of any shear stress in the flow field. In this case all the terms of σ_{ij} are neglected in the equations of motion.

Case two:

We define σ_{ij} for real viscous flows for which the shear stress is function of the fluid viscosity and the effect of shear stress in the flow field is dominant and hence can not be neglected in the equations of motion. In Part (2) of this course, we study few applications for which we solve the equations of motions for some real viscous flows.

Case one: (this case is discussed in details with examples in Part 3)

1.4 Inviscid Flow

As is discussed in 2nd year , shearing stresses develop in a moving fluid because of the viscosity of the fluid. We know that for some common fluids, such as air and water, the viscosity is small, and therefore it seems reasonable to assume that under some circumstances we may be able to simply neglect the effect of viscosity (and thus shearing stresses). Flow fields in which the shearing stresses are assumed to be negligible are said to be *inviscid*, *nonviscous*, or *frictionless*. These terms are used interchangeably. As is discussed in 2nd year , for fluids in which there are no shearing stresses the normal stress at a point is independent of direction—that is, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$. In this instance we define the pressure, p , as the negative of the normal stress so that

$$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$$

The negative sign is used so that a *compressive* normal stress (which is what we expect in a fluid) will give a *positive* value for p .

In 2nd year the inviscid flow concept was used in the development of the Bernoulli equation, and numerous applications of this important equation were considered. In this section we will again consider the Bernoulli equation and will show how it can be derived from the general equations of motion for inviscid flow.

1.4.1 Euler's Equations of Motion

For an inviscid flow in which all the shearing stresses are zero, and the normal stresses are replaced by $-p$, the general equations of motion (Eqs. 1.50) reduce to

$$\rho g_x - \frac{\partial p}{\partial x} = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \quad (1.51a)$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \quad (1.51b)$$

$$\rho g_z - \frac{\partial p}{\partial z} = \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) \quad (1.51c)$$

These equations are commonly referred to as *Euler's equations of motion*, named in honor of **Leonhard Euler** (1707–1783), a famous Swiss mathematician who pioneered work on the relationship between pressure and flow. In vector notation Euler's equations can be expressed as

$$\rho \mathbf{g} - \nabla p = \rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] \quad (1.52)$$

Although Eqs. 1.51 are considerably simpler than the general equations of motion, they are still not amenable to a general analytical solution that would allow us to determine the pressure and velocity at all points within an inviscid flow field. The main difficulty arises from the nonlinear velocity terms (such as $u \partial u / \partial x$, $v \partial u / \partial y$, etc.), which appear in the convective acceleration. Because of these terms, Euler's equations are nonlinear partial differential equations for which we do not have a general method of solving. However, under some circumstances we can use them to obtain useful information about inviscid flow fields. For example, as shown in the following section we can integrate Eq. 1.52 to obtain a relationship (the Bernoulli equation) between elevation, pressure, and velocity along a streamline.

1.4.2 The Bernoulli Equation

In 2nd year the Bernoulli equation was derived by a direct application of Newton's second law to a fluid particle moving along a streamline. In this section we will again derive this important equation, starting from Euler's equations. Of course, we should obtain the same result since Euler's equations simply represent a statement of Newton's second law expressed in a general form that is useful for flow problems. We will restrict our attention to steady flow so Euler's equation in vector form becomes

$$\rho \mathbf{g} - \nabla p = \rho (\mathbf{V} \cdot \nabla) \mathbf{V} \quad (1.53)$$

We wish to integrate this differential equation along some arbitrary streamline (Fig. 1.12) and select the coordinate system with the z axis vertical (with "up" being positive) so that the acceleration of gravity vector can be expressed as

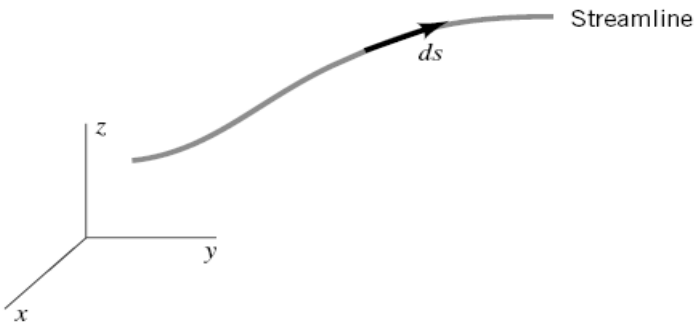
$$\mathbf{g} = -g\nabla z$$

where g is the magnitude of the acceleration of gravity vector. Also, it will be convenient to use the vector identity

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) - \mathbf{V} \times (\nabla \times \mathbf{V})$$

Equation 1.53 can now be written in the form

$$-\rho g\nabla z - \nabla p = \frac{\rho}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) - \rho\mathbf{V} \times (\nabla \times \mathbf{V})$$



■ FIGURE 1.12 The notation for differential length along a streamline.

and this equation can be rearranged to yield

$$\frac{\nabla p}{\rho} + \frac{1}{2}\nabla(V^2) + g\nabla z = \mathbf{V} \times (\nabla \times \mathbf{V})$$

We next take the dot product of each term with a differential length ds along a streamline (Fig. 1.12). Thus,

$$\frac{\nabla p}{\rho} \cdot ds + \frac{1}{2}\nabla(V^2) \cdot ds + g\nabla z \cdot ds = [\mathbf{V} \times (\nabla \times \mathbf{V})] \cdot ds \quad (1.54)$$

Since ds has a direction along the streamline, the vectors ds and \mathbf{V} are parallel. However, the vector $\mathbf{V} \times (\nabla \times \mathbf{V})$ is perpendicular to \mathbf{V} (why?), so it follows that

$$[\mathbf{V} \times (\nabla \times \mathbf{V})] \cdot ds = 0$$

Recall also that the dot product of the gradient of a scalar and a differential length gives the differential change in the scalar in the direction of the differential length. That is, with $ds = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$ we can write $\nabla p \cdot ds = (\partial p/\partial x)dx + (\partial p/\partial y)dy + (\partial p/\partial z)dz = dp$. Thus, Eq. 1.54 becomes

$$\frac{dp}{\rho} + \frac{1}{2}d(V^2) + g dz = 0 \quad (1.55)$$

where the change in p , V , and z is along the streamline. Equation 1.55 can now be integrated to give

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant} \quad (1.56)$$

which indicates that the sum of the three terms on the left side of the equation must remain a constant along a given streamline. Equation 1.56 is valid for both compressible and incompressible inviscid flows, but for compressible fluids the variation in ρ with p must be specified before the first term in Eq. 1.56 can be evaluated.

For inviscid, incompressible fluids (commonly called *ideal fluids*) Eq. 1.56 can be written as

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant} \quad (1.57)$$

and this equation is the Bernoulli equation used extensively in 2nd year . It is often convenient to write Eq. 1.57 between two points (1) and (2) along a streamline and to express the equation in the “head” form by dividing each term by g so that

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \quad (1.58)$$

It should be again emphasized that the Bernoulli equation, as expressed by Eqs. 1.57 and 1.58, is restricted to the following:

- inviscid flow
- steady flow
- incompressible flow
- flow along a streamline

Case Two: this case is discussed in details here
(in Part 2, we have examples on analysis of viscous flow)

1.8 Viscous Flow

To incorporate viscous effects into the differential analysis of fluid motion we must return to the previously derived general equations of motion, Eq. 1.50. Since these equations include both stresses and velocities, there are more unknowns than equations, and therefore before proceeding it is necessary to establish a relationship between the stresses and velocities.

1.8.1 Stress-Deformation Relationships

For incompressible Newtonian fluids it is known that the stresses are linearly related to the rates of deformation and can be expressed in Cartesian coordinates as (for normal stresses)

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} \quad (1.125a)$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y} \quad (1.125b)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z} \quad (1.125c)$$

(for shearing stresses)

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (1.125d)$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \quad (1.125e)$$

$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \quad (1.125f)$$

where p is the pressure, the negative of the average of the three normal stresses; that is $-p = (\frac{1}{3})(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$. For viscous fluids in motion the normal stresses are not necessarily the same in different directions, thus, the need to define the pressure as the average of the three normal stresses. For fluids at rest, or frictionless fluids, the normal stresses are equal in all directions. (We have made use of this fact in the chapter on fluid statics and in developing the equations for inviscid flow.) Detailed discussions of the development of these stress-velocity gradient relationships can be found in Refs. 3, 7, and 8. An important point to note is that whereas for elastic solids the stresses are linearly related to the deformation (or strain), for Newtonian fluids the stresses are linearly related to the rate of deformation (or rate of strain).

In cylindrical polar coordinates the stresses for incompressible Newtonian fluids are expressed as (for normal stresses)

$$\sigma_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r} \quad (1.126a)$$

$$\sigma_{\theta\theta} = -p + 2\mu \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \quad (1.126b)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z} \quad (1.126c)$$

(for shearing stresses)

$$\tau_{r\theta} = \tau_{\theta r} = \mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \quad (1.126d)$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right) \quad (1.126e)$$

$$\tau_{zr} = \tau_{rz} = \mu \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \quad (6.126f)$$

The double subscript has a meaning similar to that of stresses expressed in Cartesian coordinates—that is, the first subscript indicates the plane on which the stress acts, and the second subscript the direction. Thus, for example, σ_{rr} refers to a stress acting on a plane perpendicular to the radial direction and in the radial direction (thus a normal stress). Similarly, $\tau_{r\theta}$ refers to a stress acting on a plane perpendicular to the radial direction but in the tangential (θ direction) and is therefore a shearing stress.

1.8.2 The Navier–Stokes Equations

The stresses as defined in the preceding section can be substituted into the differential equations of motion (Eqs. 1.50) and simplified by using the continuity equation (Eq. 1.31) to obtain (x direction):

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (1.127a)$$

(y direction)

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \quad (1.127b)$$

(z direction)

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \quad (1.127c)$$

where we have rearranged the equations so the acceleration terms are on the left side and the force terms are on the right. These equations are commonly called the *Navier–Stokes* equations, named in honor of the French mathematician L. M. H. Navier (1758–1836) and the English mechanician Sir G. G. Stokes (1819–1903), who were responsible for their formulation. These three equations of motion, when combined with the conservation of mass equation (Eq. 1.31), provide a complete mathematical description of the flow of incompressible Newtonian fluids. We have four equations and four unknowns (u , v , w , and p), and therefore the problem is “well-posed” in mathematical terms. Unfortunately, because of the general complexity of the Navier–Stokes equations (they are nonlinear, second-order, partial differential equations), they are not amenable to exact mathematical solutions except in a few instances. However, in those few instances in which solutions have been obtained and compared with experimental results, the results have been in close agreement. Thus, the Navier–Stokes equations are considered to be the governing differential equations of motion for incompressible Newtonian fluids.

In terms of cylindrical polar coordinates (see Fig. 1.6), the Navier–Stokes equation can be written as

(r direction)

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) \\ = -\frac{\partial p}{\partial r} + \rho g_r + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_r}{\partial r} \right) - \frac{v_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] \end{aligned} \quad (1.128a)$$

(θ direction)

$$\begin{aligned} \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) \\ = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right) - \frac{v_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right] \end{aligned} \quad (1.128b)$$

(z direction)

$$\begin{aligned} \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) \\ = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] \end{aligned} \quad (1.128c)$$

To provide a brief introduction to the use of the Navier–Stokes equations, a few of the simplest exact solutions are developed in the next section. Although these solutions will prove to be relatively simple, this is not the case in general. In fact, only a few other exact solutions have been obtained.

EXAMPLE 1.4

Take the velocity field of Example 1.2C with $b = 0$ for algebraic convenience

$$u = a(x^2 - y^2) \quad v = -2axy \quad w = 0$$

and determine under what conditions it is a solution to the Navier-Stokes momentum equation (1.127). Assuming that these conditions are met, determine the resulting pressure distribution when z is “up” ($g_x = 0$, $g_y = 0$, $g_z = -g$).

Solution

Make a direct substitution of u , v , w into Eq. (1.127)

$$\rho(0) - \frac{\partial p}{\partial x} + \mu(2a - 2a) = 2a^2\rho(x^3 + xy^2) \quad (1)$$

⁴When compressibility is significant, additional small terms arise containing the element volume expansion rate and a *second* coefficient of viscosity; see Refs. 4 and 5 for details.

$$\rho(0) - \frac{\partial p}{\partial y} + \mu(0) = 2a^2\rho(x^2y + y^3) \quad (2)$$

$$\rho(-g) - \frac{\partial p}{\partial z} + \mu(0) = 0 \quad (3)$$

The viscous terms vanish identically (although μ is *not* zero). Equation (3) can be integrated partially to obtain

$$p = -\rho gz + f_1(x, y) \quad (4)$$

i.e., the pressure is hydrostatic in the z direction, which follows anyway from the fact that the flow is two-dimensional ($w = 0$). Now the question is: Do Eqs. (1) and (2) show that the given velocity field *is* a solution? One way to find out is to form the mixed derivative $\partial^2 p / (\partial x \partial y)$ from (1) and (2) separately and then compare them.

Differentiate Eq. (1) with respect to y

$$\frac{\partial^2 p}{\partial x \partial y} = -4a^2\rho xy \quad (5)$$

Now differentiate Eq. (2) with respect to x

$$\frac{\partial^2 p}{\partial x \partial y} = -\frac{\partial}{\partial x} [2a^2\rho(x^2y + y^3)] = -4a^2\rho xy \quad (6)$$

Since these are identical, the given velocity field is an *exact* solution to the Navier-Stokes

equation.

Ans.

To find the pressure distribution, substitute Eq. (4) into Eqs. (1) and (2), which will enable us to find $f_1(x, y)$

$$\frac{\partial f_1}{\partial x} = -2a^2\rho(x^3 + xy^2) \quad (7)$$

$$\frac{\partial f_1}{\partial y} = -2a^2\rho(x^2y + y^3) \quad (8)$$

Integrate Eq. (7) partially with respect to x

$$f_1 = -\frac{1}{2}a^2\rho(x^4 + 2x^2y^2) + f_2(y) \quad (9)$$

Differentiate this with respect to y and compare with Eq. (8)

$$\frac{\partial f_1}{\partial y} = -2a^2\rho x^2y + f_2'(y) \quad (10)$$

Comparing (8) and (10), we see they are equivalent if

$$f_2'(y) = -2a^2\rho y^3$$

or

$$f_2(y) = -\frac{1}{2}a^2\rho y^4 + C \quad (11)$$

where C is a constant. Combine Eqs. (4), (9), and (11) to give the complete expression for pressure distribution

$$p(x, y, z) = -\rho gz - \frac{1}{2}a^2\rho(x^4 + y^4 + 2x^2y^2) + C \quad \text{Ans.} \quad (12)$$

This is the desired solution. Do you recognize it? Not unless you go back to the beginning and square the velocity components:

$$u^2 + v^2 + w^2 = V^2 = a^2(x^4 + y^4 + 2x^2y^2) \quad (13)$$

Comparing with Eq. (12), we can rewrite the pressure distribution as

$$p + \frac{1}{2}\rho V^2 + \rho gz = C \quad (14)$$

This is Bernoulli's equation. That is no accident, because the velocity distribution given in this problem is one of a family of flows which are solutions to the Navier-Stokes equation and which satisfy Bernoulli's incompressible equation everywhere in the flow field. They are called *irrotational flows*, for which $\text{curl } \mathbf{V} = \nabla \times \mathbf{V} \equiv 0$. This subject is discussed again in Part 3

1.9.1 The Differential Equation of Angular Momentum:

Having now been through the same approach for both mass and linear momentum, we can go rapidly through a derivation of the differential angular-momentum relation. The appropriate form of the integral angular-momentum equation for a fixed control volume is

$$\sum \mathbf{M}_O = \frac{\partial}{\partial t} \left[\int_{CV} (\mathbf{r} \times \mathbf{V}) \rho \, d^3V \right] + \int_{CS} (\mathbf{r} \times \mathbf{V}) \rho (\mathbf{V} \cdot \mathbf{n}) \, dA \quad (1.129)$$

We shall confine ourselves to an axis O which is parallel to the z axis and passes through the centroid of the elemental control volume. This is shown in Fig.1.13. Let θ be the angle of rotation about O of the fluid within the control volume. The only stresses which have moments about O are the shear stresses τ_{xy} and τ_{yx} . We can evaluate the moments about O and the angular-momentum terms about O . A lot of algebra is involved, and we give here only the result

$$\begin{aligned} \left[\tau_{xy} - \tau_{yx} + \frac{1}{2} \frac{\partial}{\partial x} (\tau_{xy}) \, dx - \frac{1}{2} \frac{\partial}{\partial y} (\tau_{yx}) \, dy \right] dx \, dy \, dz \\ = \frac{1}{12} \rho (dx \, dy \, dz) (dx^2 + dy^2) \frac{d^2\theta}{dt^2} \end{aligned} \quad (1.130)$$

Assuming that the angular acceleration $d^2\theta/dt^2$ is not infinite, we can neglect all higher-

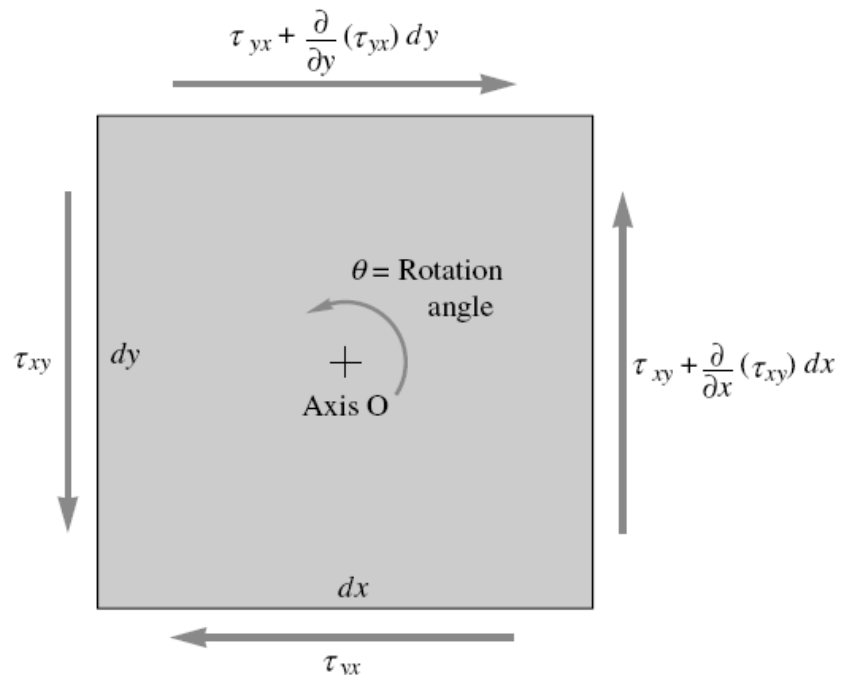


Fig.1.13 Elemental cartesian fixed control volume showing shear stresses which may cause a net angular acceleration about axis O .

order differential terms, which leaves a finite and interesting result

$$\tau_{xy} \approx \tau_{yx}$$

Had we summed moments about axes parallel to y or x , we would have obtained exactly analogous results

$$\tau_{xz} \approx \tau_{zx} \quad \tau_{yz} \approx \tau_{zy}$$

There is *no* differential angular-momentum equation. Application of the integral theorem to a differential element gives the result, well known to students of stress analysis, that the shear stresses are symmetric: $\tau_{ij} = \tau_{ji}$. This is the only result of this section.⁵ There is no differential equation to remember, which leaves room in your brain for the next topic, the differential energy equation.

1.9.2 The Differential Equation of Energy:

We are now so used to this type of derivation that we can race through the energy equation at a bewildering pace. The appropriate integral relation for the fixed control volume of Fig.1.14 is

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{\partial}{\partial t} \left(\int_{CV} e \rho dV \right) + \int_{CS} \left(e + \frac{p}{\rho} \right) \rho (\mathbf{V} \cdot \mathbf{n}) dA \quad (1.131)$$

where $\dot{W}_s = 0$ because there can be no infinitesimal shaft protruding into the control volume. By analogy with Eq. 1.129, the right-hand side becomes, for this tiny element,

$$\dot{Q} - \dot{W}_v = \left[\frac{\partial}{\partial t} (\rho e) + \frac{\partial}{\partial x} (\rho u \zeta) + \frac{\partial}{\partial y} (\rho v \zeta) + \frac{\partial}{\partial z} (\rho w \zeta) \right] dx dy dz \quad (1.132)$$

where $\zeta = e + p/\rho$. When we use the continuity equation by analogy this becomes

$$\dot{Q} - \dot{W}_v = \left(\rho \frac{de}{dt} + \mathbf{V} \cdot \nabla p \right) dx dy dz$$

To evaluate \dot{Q} , we neglect radiation and consider only heat conduction through the sides of the element. The heat flow by conduction follows Fourier's law

$$\mathbf{q} = -k \nabla T$$

where k is the coefficient of thermal conductivity of the fluid. Figure 1.14 shows the heat flow passing through the x faces, the y and z heat flows being omitted for clarity. We can list these six heat-flux terms:

Faces	Inlet heat flux	Outlet heat flux
x	$q_x dy dz$	$\left[q_x + \frac{\partial}{\partial x} (q_x) dx \right] dy dz$
y	$q_y dx dz$	$\left[q_y + \frac{\partial}{\partial y} (q_y) dy \right] dx dz$
z	$q_z dx dy$	$\left[q_z + \frac{\partial}{\partial z} (q_z) dz \right] dx dy$

⁵We are neglecting the possibility of a finite *couple* being applied to the element by some powerful external force field. See, e.g., Ref. 6, p. 217.

⁶This section may be omitted without loss of continuity.

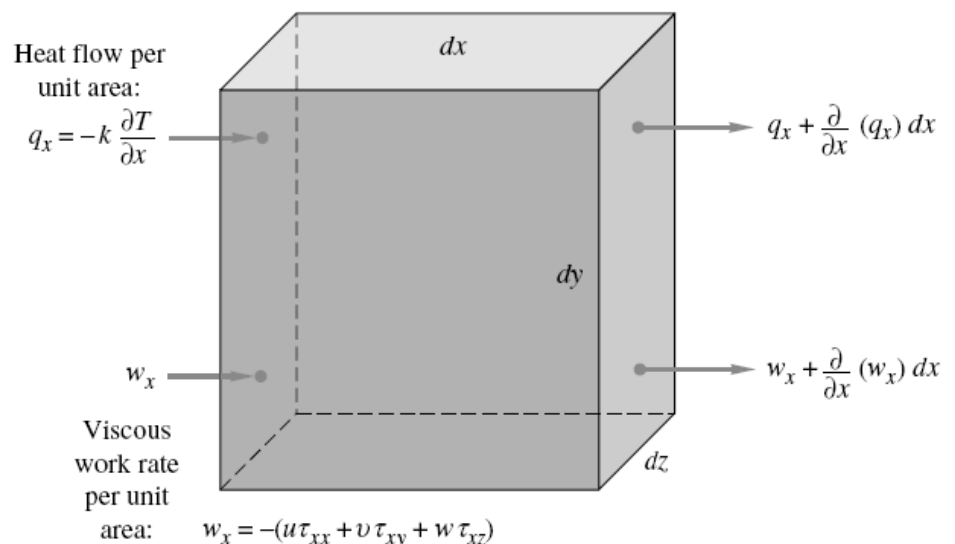


Fig.1.14 Elemental cartesian control volume showing heat-flow and viscous-work-rate terms in the x direction.

By adding the inlet terms and subtracting the outlet terms, we obtain the net heat added to the element

$$\dot{Q} = - \left[\frac{\partial}{\partial x} (q_x) + \frac{\partial}{\partial y} (q_y) + \frac{\partial}{\partial z} (q_z) \right] dx dy dz = -\nabla \cdot \mathbf{q} dx dy dz$$

As expected, the heat flux is proportional to the element volume. Introducing Fourier's law we have

$$\dot{Q} = \nabla \cdot (k \nabla T) dx dy dz$$

The rate of work done by viscous stresses equals the product of the stress component, its corresponding velocity component, and the area of the element face. Figure 1.14 shows the work rate on the left x face is

$$\dot{W}_{v,LF} = w_x dy dz \quad \text{where } w_x = -(u\tau_{xx} + v\tau_{xy} + w\tau_{xz})$$

(where the subscript LF stands for left face) and a slightly different work on the right face due to the gradient in w_x . These work fluxes could be tabulated in exactly the same manner as the heat fluxes in the previous table, with w_x replacing q_x , etc. After outlet terms are subtracted from inlet terms, the net viscous-work rate becomes

$$\begin{aligned} \dot{W}_v &= - \left[\frac{\partial}{\partial x} (u\tau_{xx} + v\tau_{xy} + w\tau_{xz}) + \frac{\partial}{\partial y} (u\tau_{yx} + v\tau_{yy} + w\tau_{yz}) \right. \\ &\quad \left. + \frac{\partial}{\partial z} (u\tau_{zx} + v\tau_{zy} + w\tau_{zz}) \right] dx dy dz \\ &= -\nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij}) dx dy dz \end{aligned}$$

We now substitute to obtain one form of the differential energy equation

$$\rho \frac{de}{dt} + \mathbf{V} \cdot \nabla p = \nabla \cdot (k \nabla T) + \nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij}) \quad \text{where } e = \hat{u} + \frac{1}{2}V^2 + gz$$

A more useful form is obtained if we split up the viscous-work term

$$\nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij}) \equiv \mathbf{V} \cdot (\nabla \cdot \boldsymbol{\tau}_{ij}) + \Phi$$

where Φ is short for the *viscous-dissipation function*.⁷ For a newtonian incompressible viscous fluid, this function has the form

$$\begin{aligned} \Phi &= \mu \left[2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial v}{\partial y} \right)^2 + 2 \left(\frac{\partial w}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right. \\ &\quad \left. + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 \right] \end{aligned}$$

Since all terms are quadratic, viscous dissipation is always positive, so that a viscous flow always tends to lose its available energy due to dissipation, in accordance with the second law of thermodynamics.

Now substitute, using the linear-momentum equation to eliminate $\nabla \cdot \boldsymbol{\tau}_{ij}$. This will cause the kinetic and potential energies to cancel, leaving a more customary form of the general differential energy equation

$$\rho \frac{d\hat{u}}{dt} + p(\nabla \cdot \mathbf{V}) = \nabla \cdot (k \nabla T) + \Phi \quad (1.133)$$

This equation is valid for a newtonian fluid under very general conditions of unsteady, compressible, viscous, heat-conducting flow, except that it neglects radiation heat transfer and internal *sources* of heat that might occur during a chemical or nuclear reaction.

Equation 1.133 is too difficult to analyze except on a digital computer [1]. It is customary to make the following approximations:

$$d\hat{u} \approx c_v dT \quad c_v, \mu, k, \rho \approx \text{const}$$

Equation 1.133 then takes the simpler form

$$\rho c_v \frac{dT}{dt} = k \nabla^2 T + \Phi$$

which involves temperature T as the sole primary variable plus velocity as a secondary variable through the total time-derivative operator

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \quad (1.134)$$

A great many interesting solutions to Eq. 1.134 are known for various flow conditions, and extended treatments are given in advanced books on viscous flow [4, 5] and books on heat transfer [7, 8].

One well-known special case of Eq. 1.134 occurs when the fluid is at rest or has negligible velocity, where the dissipation Φ and convective terms become negligible

$$\rho c_v \frac{\partial T}{\partial t} = k \nabla^2 T \quad (1.135)$$

This is called the *heat-conduction equation* in applied mathematics and is valid for solids and fluids at rest. The solution to Eq. 1.135 for various conditions is a large part of courses and books on heat transfer.

This completes the derivation of the basic differential equations of fluid motion.

⁷For further details, see, e.g., Ref. 5, p. 72.

1.9.4 Boundary Conditions for the Basic Differential Equations:

There are three basic differential equations of fluid motion, just derived. Let us summarize them here:

Continuity:
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \quad (1.136)$$

Momentum:
$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\tau}_{ij} \quad (1.137)$$

Energy:
$$\rho \frac{d\hat{u}}{dt} + p(\nabla \cdot \mathbf{V}) = \nabla \cdot (k \nabla T) + \Phi \quad (1.138)$$

where Φ is given in eq. 1.133. In general, the density is variable, so that these three equations contain five unknowns, ρ , V , p , \hat{u} , and T . Therefore we need two additional relations to complete the system of equations. These are provided by data or algebraic expressions for the state relations of the thermodynamic properties

$$\rho = \rho(p, T) \quad \hat{u} = \hat{u}(p, T) \quad (1.139)$$

For example, for a perfect gas with constant specific heats, we complete the system with

$$\rho = \frac{p}{RT} \quad \hat{u} = \int c_v dT \approx c_v T + \text{const} \quad (1.140)$$

It is shown in advanced books [4, 5] that this system of equations 1.136 to 1.139 is well posed and can be solved analytically or numerically, subject to the proper boundary conditions.

What are the proper boundary conditions? First, if the flow is unsteady, there must be an *initial condition* or initial spatial distribution known for each variable:

At $t = 0$: $\rho, V, p, \hat{u}, T = \text{known } f(x, y, z)$

Thereafter, for all times t to be analyzed, we must know something about the variables at each *boundary* enclosing the flow.

Figure 1.15 illustrates the three most common types of boundaries encountered in fluid-flow analysis: a solid wall, an inlet or outlet, a liquid-gas interface.

First, for a solid, impermeable wall, there is no slip and no temperature jump in a viscous heat-conducting fluid

$$\mathbf{V}_{\text{fluid}} = \mathbf{V}_{\text{wall}} \quad T_{\text{fluid}} = T_{\text{wall}} \quad \text{solid wall}$$

The only exception to last eq. occurs in an extremely rarefied gas flow, where slippage can be present [5].

Second, at any inlet or outlet section of the flow, the complete distribution of velocity, pressure, and temperature must be known for all times:

Inlet or outlet: $\text{Known } \mathbf{V}, p, T$

These inlet and outlet sections can be and often are at $\pm \infty$, simulating a body immersed in an infinite expanse of fluid.

Finally, the most complex conditions occur at a liquid-gas interface, or free surface, as sketched in Fig. 1.15. Let us denote the interface by

Interface:

$$z = \eta(x, y, t)$$

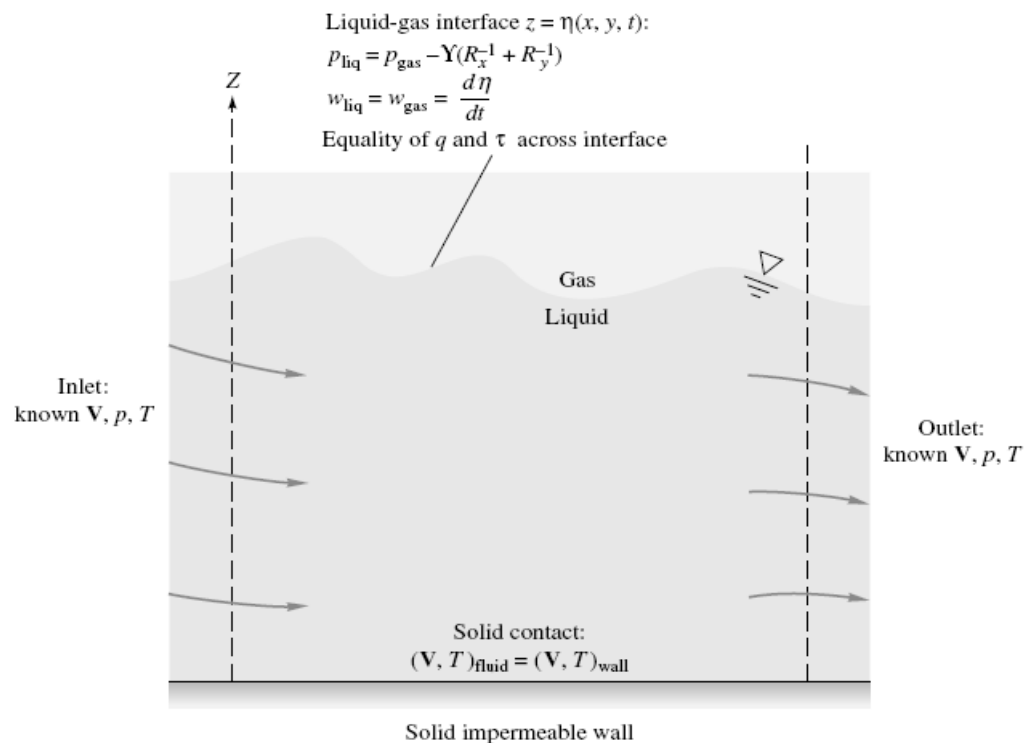


Fig. 1.15 Typical boundary conditions in a viscous heat-conducting fluid-flow analysis.

Then there must be equality of vertical velocity across the interface, so that no holes appear between liquid and gas:

$$w_{\text{liq}} = w_{\text{gas}} = \frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y}$$

This is called the *kinematic boundary condition*.

There must be mechanical equilibrium across the interface. The viscous-shear stresses must balance

$$(\tau_{zy})_{\text{liq}} = (\tau_{zy})_{\text{gas}} \quad (\tau_{zx})_{\text{liq}} = (\tau_{zx})_{\text{gas}}$$

Neglecting the viscous normal stresses, the pressures must balance at the interface except for surface-tension effects

$$p_{\text{liq}} = p_{\text{gas}} - \Upsilon(R_x^{-1} + R_y^{-1})$$

which is equivalent to Eq. 1.136. The radii of curvature can be written in terms of the free-surface position η

$$R_x^{-1} + R_y^{-1} = \frac{\partial}{\partial x} \left[\frac{\partial\eta/\partial x}{\sqrt{1 + (\partial\eta/\partial x)^2 + (\partial\eta/\partial y)^2}} \right] + \frac{\partial}{\partial y} \left[\frac{\partial\eta/\partial y}{\sqrt{1 + (\partial\eta/\partial x)^2 + (\partial\eta/\partial y)^2}} \right]$$

Finally, the heat transfer must be the same on both sides of the interface, since no heat can be stored in the infinitesimally thin interface

$$(q_z)_{\text{liq}} = (q_z)_{\text{gas}}$$

Neglecting radiation, this is equivalent to

$$\left(k \frac{\partial T}{\partial z} \right)_{\text{liq}} = \left(k \frac{\partial T}{\partial z} \right)_{\text{gas}}$$

This is as much detail as we wish to give at this level of exposition. Further and even more complicated details on fluid-flow boundary conditions are given in Refs. 5 and 9.

Simplified Free Surface Conditions:

In the introductory analyses given in this book, such as open-channel flows in Chap. 10, we shall back away from the exact conditions shown before and assume that the upper fluid is an “atmosphere” which merely exerts pressure upon the lower fluid, with shear and heat conduction negligible. We also neglect nonlinear terms involving the slopes of the free surface. We then have a much simpler and linear set of conditions at the surface

$$p_{\text{liq}} \approx p_{\text{gas}} - \Upsilon \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) \quad w_{\text{liq}} \approx \frac{\partial \eta}{\partial t}$$

$$\left(\frac{\partial V}{\partial z} \right)_{\text{liq}} \approx 0 \quad \left(\frac{\partial T}{\partial z} \right)_{\text{liq}} \approx 0$$

In many cases, such as open-channel flow, we can also neglect surface tension, so that

$$p_{\text{liq}} \approx p_{\text{atm}}$$

These are the types of approximations which will be used in Chap. 10. The nondimensional forms of these conditions will also be useful in Chap. 5.

Incompressible flow with constant properties:

Flow with constant ρ , μ , and k is a basic simplification which will be used, e.g., throughout Chap. 6. The basic equations of motion 1.136-1.138 reduce to:

Continuity: $\nabla \cdot \mathbf{V} = 0$

Momentum: $\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{V}$

Energy: $\rho c_v \frac{dT}{dt} = k \nabla^2 T + \Phi$

Since ρ is constant, there are only three unknowns: p , \mathbf{V} , and T . The system is closed.⁸ Not only that, the system splits apart: Continuity and momentum are independent of T . Thus we can solve the above eqs. entirely separately for the pressure and velocity, using such boundary conditions as

Solid surface: $\mathbf{V} = \mathbf{V}_{\text{wall}}$

⁸For this system, what are the thermodynamic equivalents to Eq. (4.59)?

$$v_z = U \left(1 - \frac{r^2}{R^2} \right) \quad v_r = 0 \quad v_\theta = 0$$

where U is the maximum or centerline velocity and R is the tube radius. If the wall temperature is constant at T_w and the temperature $T = T(r)$ only, find $T(r)$ for this flow.

For the given conditions, the energy equation reduces to

$$\rho C_v v_r \frac{dT}{dr} = \frac{k}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \mu \left(\frac{dv_z}{dr} \right)^2$$

Substituting for v_z and realizing the $v_r = 0$, we obtain

Inlet or outlet: Known \mathbf{V} , p

Free surface: $p \approx p_a \quad w \approx \frac{\partial \eta}{\partial t}$

Later, entirely at our leisure,⁹ we can solve for the temperature distribution from Eq. , which depends upon velocity \mathbf{V} through the dissipation Φ and the total time-derivative operator d/dt .

⁹Since temperature is entirely *uncoupled* by this assumption, we may never get around to solving for it here and may ask you to wait until a course on heat transfer.

Frictionless Flow Approximation:

Chapter 8 assumes inviscid flow throughout, for which the viscosity $\mu = 0$. The momentum equation reduces to

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p$$

This is *Euler's equation*; it can be integrated along a streamline to obtain Bernoulli's equation (see Sec.). By neglecting viscosity we have lost the second-order derivative of \mathbf{V} in Eq. ; therefore we must relax one boundary condition on velocity. The only mathematically sensible condition to drop is the no-slip condition at the wall. We let the flow slip parallel to the wall but do not allow it to flow into the wall. The proper inviscid condition is that the normal velocities must match at any solid surface:

Inviscid flow: $(V_n)_{\text{fluid}} = (V_n)_{\text{wall}}$

In most cases the wall is fixed; therefore the proper inviscid-flow condition is

$$V_n = 0$$

There is *no* condition whatever on the tangential velocity component at the wall in inviscid flow. The tangential velocity will be part of the solution, and the correct value will appear after the analysis is completed (see Chap. 8).

Example 1.5 on Solving the Energy Equation Analytically:

$$\frac{k}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) = - \mu \left(\frac{dv_z}{dr} \right)^2 = - \frac{4U^2 \mu r^2}{R^4}$$

Multiply by r/k and integrate to obtain

$$\frac{dT}{dr} = -\frac{\mu U^2 r^3}{k R^4} + C_1$$

Integrate a second time to obtain

$$T = -\frac{\mu U^2 r^4}{4 k R^4} + C_1 \ln r + C_2$$

Since the term, $\ln r$, approaches infinity as r approaches 0, $C_1 = 0$.

Applying the wall boundary condition, $T = T_w$ at $r = R$, we obtain for C_2

$$C_2 = T_w + \frac{\mu U^2}{4 k}$$

The final solution then becomes

$$T(r) = T_w + \frac{\mu U^2}{4 k} \left(1 - \frac{r^4}{R^4}\right)$$

1.9.5 Summary of the equations of motion and energy in Cylindrical Coordinates:

The equations of motion of an incompressible newtonian fluid with constant μ , k , and c_p are given here in cylindrical coordinates (r, θ, z) , which are related to cartesian coordinates (x, y, z) as in Fig. 4.2:

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z \quad (D.1)$$

The velocity components are v_r , v_θ , and v_z . The equations are:

Continuity:

$$\frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (v_\theta) + \frac{\partial}{\partial z} (v_z) = 0 \quad (D.2)$$

Convective time derivative:

$$\mathbf{V} \cdot \nabla = v_r \frac{\partial}{\partial r} + \frac{1}{r} v_\theta \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \quad (D.3)$$

Laplacian operator:

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad (D.4)$$

The r -momentum equation:

$$\frac{\partial v_r}{\partial t} + (\mathbf{V} \cdot \nabla) v_r - \frac{1}{r} v_\theta^2 = -\frac{1}{\rho} \frac{\partial p}{\partial r} + g_r + \nu \left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right) \quad (D.5)$$

The θ -momentum equation:

$$\frac{\partial v_\theta}{\partial t} + (\mathbf{V} \cdot \nabla) v_\theta + \frac{1}{r} v_r v_\theta = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + g_\theta + \nu \left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right) \quad (D.6)$$

The z -momentum equation:

$$\frac{\partial v_z}{\partial t} + (\mathbf{V} \cdot \nabla) v_z = -\frac{1}{\rho} \frac{\partial p}{\partial z} + g_z + \nu \nabla^2 v_z \quad (D.7)$$

The energy equation:

$$\rho c_p \left[\frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla) T \right] = k \nabla^2 T + \mu [2(\epsilon_{rr}^2 + \epsilon_{\theta\theta}^2 + \epsilon_{zz}^2) + \epsilon_{\theta z}^2 + \epsilon_{rz}^2 + \epsilon_{r\theta}^2] \quad (\text{D.8})$$

where

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial v_r}{\partial r} & \epsilon_{\theta\theta} &= \frac{1}{r} \left(\frac{\partial v_\theta}{\partial \theta} + v_r \right) \\ \epsilon_{zz} &= \frac{\partial v_z}{\partial z} & \epsilon_{\theta z} &= \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \\ \epsilon_{rz} &= \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \epsilon_{r\theta} &= \frac{1}{r} \left(\frac{\partial v_r}{\partial \theta} - v_\theta \right) + \frac{\partial v_\theta}{\partial r} \end{aligned} \quad (\text{D.9})$$

Viscous stress components:

$$\begin{aligned} \tau_{rr} &= 2\mu\epsilon_{rr} & \tau_{\theta\theta} &= 2\mu\epsilon_{\theta\theta} & \tau_{zz} &= 2\mu\epsilon_{zz} \\ \tau_{r\theta} &= \mu\epsilon_{r\theta} & \tau_{\theta z} &= \mu\epsilon_{\theta z} & \tau_{rz} &= \mu\epsilon_{rz} \end{aligned} \quad (\text{D.10})$$

Angular-velocity components:

$$\begin{aligned} \omega_r &= \frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \\ \omega_\theta &= \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \\ \omega_z &= \frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \end{aligned} \quad (\text{D.11})$$

1.10 Other Aspects of Differential Analysis

In this chapter the basic differential equations that govern the flow of fluids have been developed. The Navier–Stokes equations, which can be compactly expressed in vector notation as

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V} \quad (\text{1.141})$$

along with the continuity equation

$$\nabla \cdot \mathbf{V} = 0 \quad (\text{1.142})$$

are the general equations of motion for incompressible Newtonian fluids. Although we have restricted our attention to incompressible fluids, these equations can be readily extended to include compressible fluids. It is well beyond the scope of this introductory text to consider in depth the variety of analytical and numerical techniques that can be used to obtain both exact and approximate solutions to the Navier–Stokes equations. Students, however, should be aware of the existence of these very general equations, which are frequently used as the basis for many advanced analyses of fluid motion. A few relatively simple solutions have been obtained and discussed in this chapter to indicate the type of detailed flow information that can be obtained by using differential analysis. However, it is hoped that the relative ease with which these solutions were obtained does not give the false impression that solutions to the Navier–Stokes equations are readily available. This is certainly not true, and as previously mentioned there are actually very few practical fluid flow problems that can be solved by using an exact analytical approach. In fact, there are no known analytical solutions to Eq. 1.141 for flow past any object such as a sphere, cube, or airplane.

Because of the difficulty in solving the Navier–Stokes equations, much attention has been given to various types of approximate solutions. For example, if the viscosity is set equal to zero, the Navier–Stokes equations reduce to Euler’s equations. Thus, the frictionless fluid solutions discussed previously are actually approximate solutions to the Navier–Stokes equations. At the other extreme, for problems involving slowly moving fluids, viscous effects may be dominant and the nonlinear (convective) acceleration terms can be neglected.

This assumption greatly simplifies the analysis, since the equations now become linear. There are numerous analytical solutions to these “*slow flow*” or “*creeping flow*” problems. Another broad class of approximate solutions is concerned with flow in the very thin boundary layer. L. Prandtl showed in 1904 how the Navier–Stokes equations could be simplified to study flow in boundary layers. Such “boundary layer solutions” play a very important role in the study of fluid mechanics. A further discussion of boundary layers is given in Part 4.

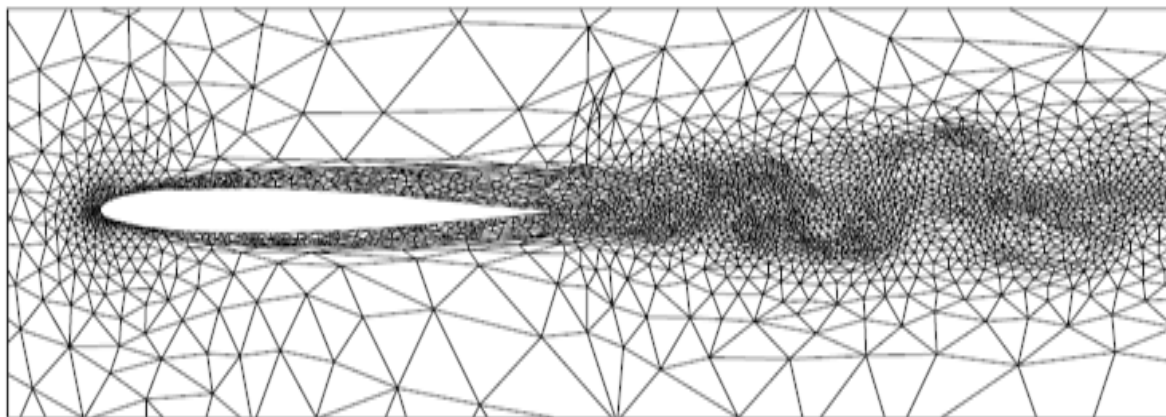
1.10.1 Numerical Methods

Numerical methods using digital computers are, of course, commonly utilized to solve a wide variety of flow problems. As discussed previously, although the differential equations that govern the flow of Newtonian fluids [the Navier–Stokes equations (1.141)] were derived many years ago, there are few known analytical solutions to them. With the advent of high-speed digital computers it has become possible to obtain approximate numerical solutions to these (and other fluid mechanics) equations for a wide variety of circumstances.

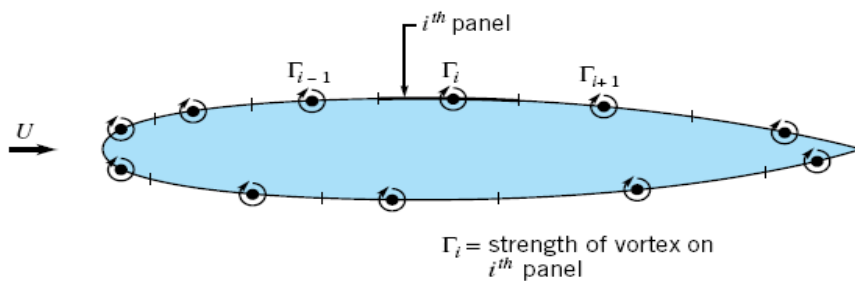
Of the various techniques available for the numerical solution of the governing differential equations of fluid flow, the following three types are most common: (1) the finite difference method, (2) the finite element (or finite volume) method, and (3) the boundary element method. In each of these methods the continuous flow field (i.e., velocity or pressure as a function of space and time) is described in terms of discrete (rather than continuous) values at prescribed locations. By this technique the differential equations are replaced by a set of algebraic equations that can be solved on the computer.

For the finite element (or finite volume) method, the flow field is broken into a set of small fluid elements (usually triangular areas if the flow is two-dimensional, or small volume elements if the flow is three-dimensional). The conservation equations (i.e., conservation of mass, momentum, and energy) are written in an appropriate form for each element, and the set of resulting algebraic equations is solved numerically for the flow field. The number, size, and shape of the elements are dictated in part by the particular flow geometry and flow conditions for the problem at hand. As the number of elements increases (as is necessary for flows with complex boundaries), the number of simultaneous algebraic equations that must be solved increases rapidly. Problems involving 1000 to 10,000 elements and 50,000 equations are not uncommon. A mesh for calculating flow past an airfoil is shown in Fig. 1.16. Further information about this method can be found in Refs. 10 and 13.

For the boundary element method, the boundary of the flow field (not the entire flow field as in the finite element method) is broken into discrete segments (Ref. 14), and appropriate singularities such as sources, sinks, doublets, and vortices are distributed on these



■ **FIGURE 1.16** Anisotropic adaptive mesh for the calculation of viscous flow over a NACA 0012 airfoil at a Reynolds number of 10,000, Mach number of 0.755, and angle of attack of 1.5° . (From CFD Laboratory, Concordia University, Montreal, Canada. Used by permission.)



■ **FIGURE 1.17**
Panel method for flow
past an airfoil.

boundary elements. The strength and type of the singularities are chosen so that the appropriate boundary conditions of the flow are obtained on the boundary elements. For points in the flow field not on the boundary, the flow is calculated by adding the contributions from the various singularities on the boundary. Although the details of this method are rather mathematically sophisticated, it may (depending on the particular problem) require less computational time and space than the finite element method.

Typical boundary elements and their associated singularities (vortices) for two-dimensional flow past an airfoil are shown in Fig. 1.17. Such use of the boundary element method in aerodynamics is often termed the *panel method* in recognition of the fact that each element plays the role of a panel on the airfoil surface (Ref. 15).

The finite difference method for computational fluid dynamics is perhaps the most easily understood and widely used of the three methods listed above. For this method the flow field is dissected into a set of grid points and the continuous functions (velocity, pressure, etc.) are approximated by discrete values of these functions calculated at the grid points. Derivatives of the functions are approximated by using the differences between the function values at neighboring grid points divided by the grid spacing. The differential equations are thereby transferred into a set of algebraic equations, which is solved by appropriate numerical techniques. The larger the number of grid points used, the larger the number of equations that must be solved. It is usually necessary to increase the number of grid points (i.e., use a finer mesh) where large gradients are to be expected, such as in the boundary layer near a solid surface.

A very simple one-dimensional example of the finite difference technique is presented in the following example.

A Solved Example 1.6 on Numerical Solutions:

A viscous oil flows from a large, open tank and through a long, small-diameter pipe as shown in Fig. E 1.11a. At time $t = 0$ the fluid depth is H . Use a finite difference technique to determine the liquid depth as a function of time, $h = h(t)$. Compare this result with the exact solution of the governing equation.

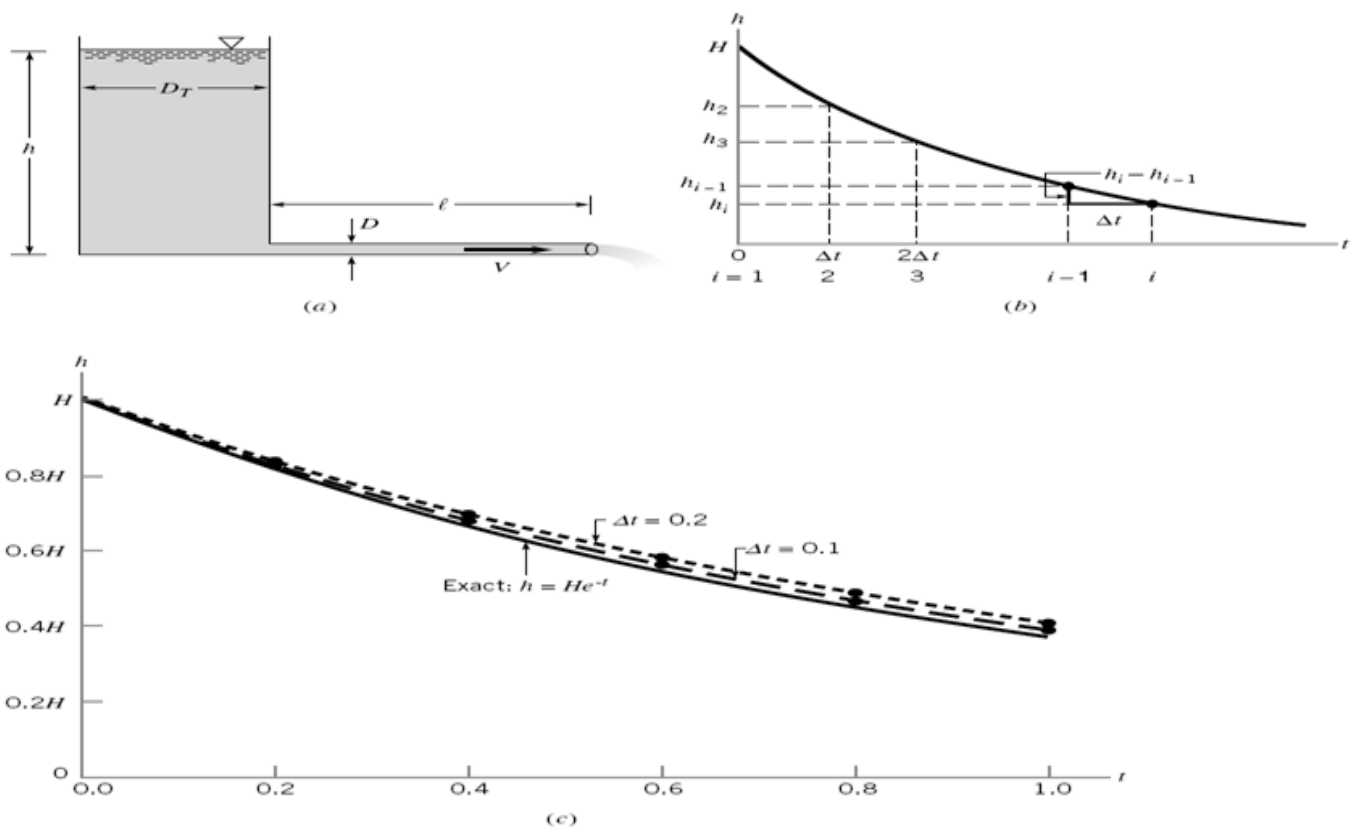
SOLUTION

Although this is an unsteady flow (i.e., the deeper the oil, the faster it flows from the tank) we assume that the flow is “quasisteady” (assume steady) and apply steady flow equations as follows.

As shown later in part 2, the mean velocity, V , for steady laminar flow in a round pipe of diameter D is given by

$$V = \frac{D^2 \Delta p}{32 \mu \ell} \quad (1)$$

where Δp is the pressure drop over the length ℓ . For this problem the pressure at the bottom of the tank (the inlet of the pipe) is γh and that at the pipe exit is zero. Hence, $\Delta p = \gamma h$ and



■ FIGURE E 1.11

Eq. 1 becomes

$$V = \frac{D^2 \gamma h}{32 \mu \ell} \quad (2)$$

Conservation of mass requires that the flowrate from the tank, $Q = \pi D^2 V / 4$, is related to the rate of change of depth of oil in the tank, dh/dt , by

$$Q = -\frac{\pi}{4} D_T^2 \frac{dh}{dt}$$

where D_T is the tank diameter. Thus,

$$\frac{\pi}{4} D^2 V = -\frac{\pi}{4} D_T^2 \frac{dh}{dt}$$

or

$$V = -\left(\frac{D_T}{D}\right)^2 \frac{dh}{dt} \quad (3)$$

By combining Eqs. 2 and 3 we obtain

$$\frac{D^2 \gamma h}{32 \mu \ell} = -\left(\frac{D_T}{D}\right)^2 \frac{dh}{dt}$$

or

$$\frac{dh}{dt} = -Ch$$

where $C = \gamma D^4 / 32 \mu \ell D_T^2$ is a constant. For simplicity we assume the conditions are such that $C = 1$. Thus, we must solve

$$\frac{dh}{dt} = -h \quad \text{with} \quad h = H \text{ at } t = 0 \quad (4)$$

The exact solution to Eq. 4 is obtained by separating the variables and integrating to obtain

$$h = He^{-t} \quad (5)$$

However, assume this solution were not known. The following finite difference technique can be used to obtain an approximate solution.

As shown in Fig. E1.11b, we select discrete points (nodes or grid points) in time and approximate the time derivative of h by the expression

$$\left. \frac{dh}{dt} \right|_{t=t_i} \approx \frac{h_i - h_{i-1}}{\Delta t} \quad (6)$$

where Δt is the time step between the different node points on the time axis and h_i and h_{i-1} are the approximate values of h at nodes i and $i - 1$. Equation 6 is called the backward-difference approximation to dh/dt . We are free to select whatever value of Δt that we wish. (Although we do not need to space the nodes at equal distances, it is often convenient to do so.) Since the governing equation (Eq. 4) is an ordinary differential equation, the “grid” for the finite difference method is a one-dimensional grid as shown in Fig. E1.11b rather than a two-dimensional grid (which occurs for partial differential equations) as shown in Fig. E1.12b.

Thus, for each value of $i = 2, 3, 4, \dots$ we can approximate the governing equation, Eq. 4, as

$$\frac{h_i - h_{i-1}}{\Delta t} = -h_i$$

or

$$h_i = \frac{h_{i-1}}{(1 + \Delta t)} \quad (7)$$

We cannot use Eq. 7 for $i = 1$ since it would involve the nonexisting h_0 . Rather we use the initial condition (Eq. 4), which gives

$$h_1 = H$$

The result is the following set of N algebraic equations for the N approximate values of h at times $t_1 = 0, t_2 = \Delta t, \dots, t_N = (N - 1)\Delta t$.

$$\begin{aligned} h_1 &= H \\ h_2 &= h_1/(1 + \Delta t) \\ h_3 &= h_2/(1 + \Delta t) \\ &\vdots \\ h_N &= h_{N-1}/(1 + \Delta t) \end{aligned}$$

For most problems the corresponding equations would be more complicated than those just given, and a computer would be used to solve for the h_i . For this problem the solution is simply

$$\begin{aligned} h_2 &= H/(1 + \Delta t) \\ h_3 &= H/(1 + \Delta t)^2 \\ &\vdots \end{aligned}$$

or in general

$$h_i = H/(1 + \Delta t)^{i-1}$$

The results for $0 < t < 1$ are shown in Fig. E1.11c. Tabulated values of the depth for $t = 1$ are listed in the table below.

Δt	i for $t = 1$	h_i for $t = 1$
0.2	6	0.4019H
0.1	11	0.3855H
0.01	101	0.3697H
0.001	1001	0.3681H
Exact (Eq. 4)	—	0.3678H

It is seen that the approximate results compare quite favorably with the exact solution given by Eq. 5. It is expected that the finite difference results would more closely approximate the exact results as Δt is decreased since in the limit of $\Delta t \rightarrow 0$ the finite difference approximation for the derivatives (Eq. 6) approaches the actual definition of the derivative.

1st Fluid Report

- 1 أجب بـ١٢سؤال عن الأسئلة التالية (التقارير والإجابات المنقولة أو المصورة درجتها ستكون صفر ولن تُقبل).
- 2 إجابة كل سؤال تكون على صفحة مستقلة ولا تزيد عن صفحة واحدة.
- 3 تقديم التقارير في المحاضرة أو حتى الساعة 12 (مكتب د. محسن سليمان في مبنى ميكانيكا السيارات).

- 1-Engineering students may study Fluid Mechanics for many different reasons. Discuss two different reasons for studying Fluid Mechanics in Mechanical Power Engineering.
- 2-We can study Fluid Mechanics using (a) Differential method; or (b) Integral method; or (c) Dimensional analysis with some experimental work.
Explain very briefly those methods showing the main differences between them regarding the reason for and the output result of each method. Give an example for each method.
- 3-What is the viscosity of a fluid? Why viscosity is important in some flows and is not important in other flows (give some examples for each case)? Discuss how is Reynolds Number a measure to if viscous effects are important or not in any flow? Give some examples to support your discussion.
- 4-Define both the No-slip condition and the stream-line in a flow? What is the difference between Newtonian fluids and Non-newtonian fluids? Give some examples for each type.
- 5-What do you know about the conservation equations in Fluid Mechanics? Stat three main conservation equations of Fluid Mechanics.

6- Correct each of the following statement:

- a- By solving the conservation equations in the integral form for a an integral control volume, we can get exact and very detailed equations for velocity and pressure in the flow field.
- b- The partial differential momentum equations for a fluid particle are 1st order and linear equations and therefore are easy to be solved for any geometric flow field or by using a small computer.
- c- Navier-Stoke's equations represent the conservation of energy for a fluid particle in a nonviscous (frictionless) flow and can be used for Newtonian fluids only.
- d- Euler's equations represent the linear momentum conservation equations for a laminar flow where the no-slip condition must be valied far from the wall.
- e- Euler's equations can not be solved anywhere in a real viscous flow in a pipe especially at the wall of the pipe or at the center-line where internal friction is negligible.

Word Problems

- 1 The total acceleration of a fluid particle is given by Eq. () in the eulerian system, where \mathbf{V} is a known function of space and time. Explain how we might evaluate particle acceleration in the lagrangian frame, where particle position \mathbf{r} is a known function of time and initial position, $\mathbf{r} = \text{fcn}(\mathbf{r}_0, t)$. Can you give an illustrative example?
- 2 Is it true that the continuity relation, Eq. (), is valid for both viscous and inviscid, newtonian and nonnewtonian, compressible and incompressible flow? If so, are there *any* limitations on this equation?
- 3 Consider a CD compact disk rotating at angular velocity Ω . Does it have *vorticity* in the sense of this chapter? If so, how much vorticity?
- 4 How much acceleration can fluids endure? Are fluids like astronauts, who feel that $5g$ is severe? Perhaps use the flow pattern of Example (), at $r = R$, to make some estimates of fluid-acceleration magnitudes.
- 5 State the conditions (there are more than one) under which the analysis of temperature distribution in a flow field can be completely uncoupled, so that a separate analysis for velocity and pressure is possible. Can we do this for both laminar and turbulent flow?
- 6 Consider liquid flow over a dam or weir. How might the boundary conditions and the flow pattern change when we compare water flow over a large prototype to SAE 30 oil flow over a tiny scale model?
- 7 What is the difference between the stream function ψ and our method of finding the streamlines from Sec. ()? Or are they essentially the same?
- 8 Under what conditions do both the stream function ψ and the velocity potential ϕ exist for a flow field? When does one exist but not the other?

Problems:

- i) Given the steady, incompressible velocity distribution $\mathbf{V} = 3x\mathbf{i} + Cy\mathbf{j} + 0\mathbf{k}$, where C is a constant, if conservation of mass is satisfied, the value of C should be
(a) 3, (b) 3/2, (c) 0, (d) -3/2, (e) -3
- ii) Given the steady velocity distribution $\mathbf{V} = 3x\mathbf{i} + 0\mathbf{j} + Cy\mathbf{k}$, where C is a constant, if the flow is irrotational, the value of C should be
(a) 3, (b) 3/2, (c) 0, (d) -3/2, (e) -3
- iii) Given the steady, incompressible velocity distribution $\mathbf{V} = 3x\mathbf{i} + Cy\mathbf{j} + 0\mathbf{k}$, where C is a constant, the shear stress τ_{xy} at the point (x, y, z) is given by
(a) 3μ , (b) $(3x + Cy)\mu$, (c) 0, (d) $C\mu$, (e) $(3 + C)\mu$

- 1 An idealized velocity field is given by the formula

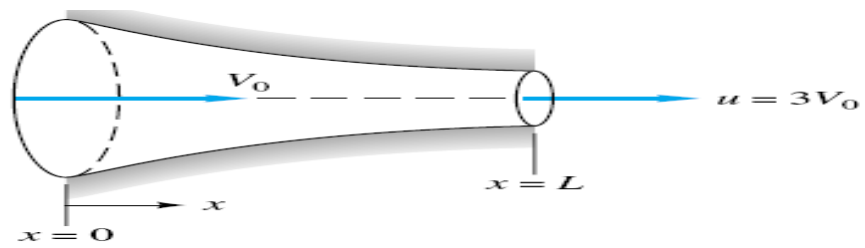
$$\mathbf{V} = 4tx\mathbf{i} - 2t^2y\mathbf{j} + 4xz\mathbf{k}$$

Is this flow field steady or unsteady? Is it two- or three-dimensional? At the point $(x, y, z) = (-1, 1, 0)$, compute (a) the acceleration vector and (b) any unit vector normal to the acceleration.

- 2 Flow through the converging nozzle in Fig. P .2 can be approximated by the one-dimensional velocity distribution

$$u \approx V_0 \left(1 + \frac{2x}{L} \right) \quad v \approx 0 \quad w \approx 0$$

(a) Find a general expression for the fluid acceleration in the nozzle. (b) For the specific case $V_0 = 10$ ft/s and $L = 6$ in, compute the acceleration, in g 's, at the entrance and at the exit.



P 2

- 3 A two-dimensional velocity field is given by

$$\mathbf{V} = (x^2 - y^2 + x)\mathbf{i} - (2xy + y)\mathbf{j}$$

in arbitrary units. At $(x, y) = (1, 2)$, compute (a) the accelerations a_x and a_y , (b) the velocity component in the direction $\theta = 40^\circ$, (c) the direction of maximum velocity, and (d) the direction of maximum acceleration.

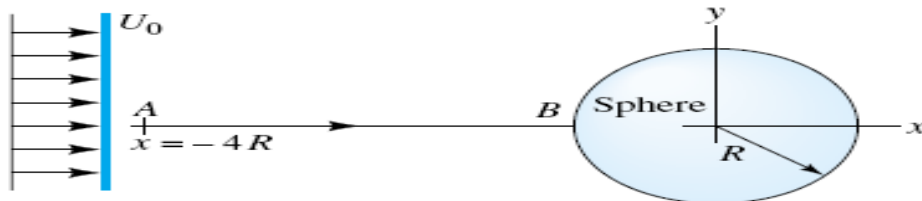
- 4 Suppose that the temperature field $T = 4x^2 - 3y^3$, in arbitrary units, is associated with the velocity field of Prob. 3. Compute the rate of change dT/dt at $(x, y) = (2, 1)$.
- 5 The velocity field near a stagnation point (see Example 1.10) may be written in the form

$$u = \frac{U_0 x}{L} \quad v = -\frac{U_0 y}{L} \quad U_0 \text{ and } L \text{ are constants}$$

- (a) Show that the acceleration vector is purely radial. (b) For the particular case $L = 1.5$ m, if the acceleration at $(x, y) = (1 \text{ m}, 1 \text{ m})$ is 25 m/s^2 , what is the value of U_0 ?
- 6 Assume that flow in the converging nozzle of Fig. P 2 has the form $\mathbf{V} = V_0[1 + (2x)/L]\mathbf{i}$. Compute (a) the fluid acceleration at $x = L$ and (b) the time required for a fluid particle to travel from $x = 0$ to $x = L$.
- 7 Consider a sphere of radius R immersed in a uniform stream U_0 , as shown in Fig. P 7. According to the theory of Chap. 8, the fluid velocity along streamline AB is given by

$$\mathbf{V} = u\mathbf{i} = U_0\left(1 + \frac{R^3}{x^3}\right)\mathbf{i}$$

Find (a) the position of maximum fluid acceleration along AB and (b) the time required for a fluid particle to travel from A to B .

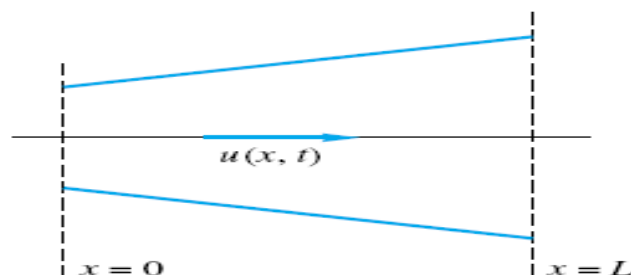


P 7

- 8 When a valve is opened, fluid flows in the expansion duct of Fig. according to the approximation

$$\mathbf{V} = \mathbf{i}U\left(1 - \frac{x}{2L}\right) \tanh \frac{Ut}{L}$$

Find (a) the fluid acceleration at $(x, t) = (L, L/U)$ and (b)

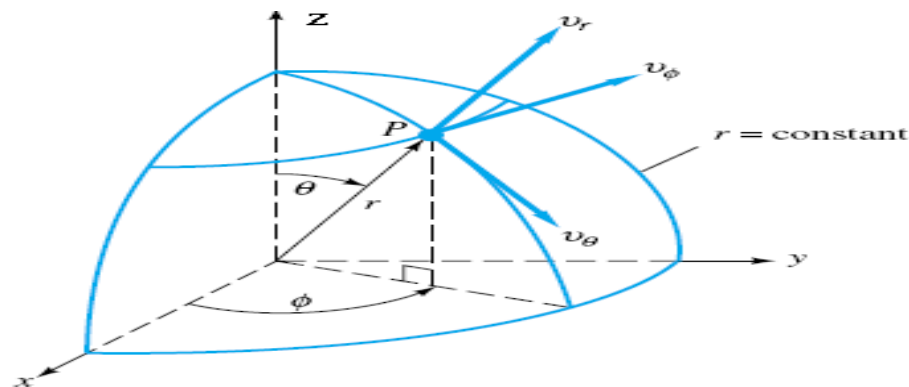


P 8

the time for which the fluid acceleration at $x = L$ is zero. Why does the fluid acceleration become negative after condition (b)?

- 9 A velocity field is given by $\mathbf{V} = (3y^2 - 3x^2)\mathbf{i} + Cxy\mathbf{j} + 0\mathbf{k}$. Determine the value of the constant C if the flow is to be (a) incompressible and (b) irrotational.
- 10 Write the special cases of the equation of continuity for (a) steady compressible flow in the yz plane, (b) unsteady incompressible flow in the xz plane, (c) unsteady compressible flow in the y direction only, (d) steady compressible flow in plane polar coordinates.
- 11 Derive Eq. () for cylindrical coordinates by considering the flux of an incompressible fluid in and out of the elemental control volume in Fig. .
- 12 Spherical polar coordinates (r, θ, ϕ) are defined in Fig. P 12. The cartesian transformations are

$$\begin{aligned}x &= r \sin \theta \cos \phi \\y &= r \sin \theta \sin \phi \\z &= r \cos \theta\end{aligned}$$



P 12

The cartesian incompressible continuity relation () can be transformed to the spherical polar form

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (v_\phi) = 0$$

What is the most general form of v_r when the flow is purely radial, that is, v_θ and v_ϕ are zero?

- 13 A two-dimensional velocity field is given by

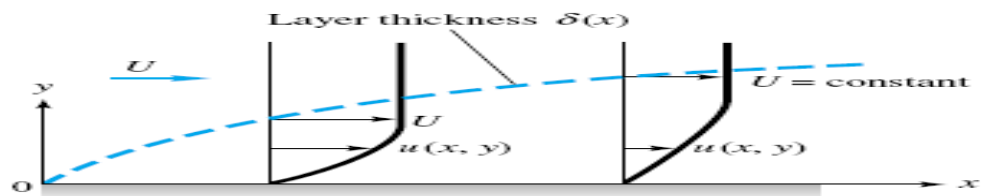
$$u = -\frac{Ky}{x^2 + y^2} \quad v = \frac{Kx}{x^2 + y^2}$$

where K is constant. Does this field satisfy incompressible continuity? Transform these velocities to polar components v_r and v_θ . What might the flow represent?

- 14 For incompressible polar-coordinate flow, what is the most general form of a purely circulatory motion, $v_\theta = v_\theta(r, \theta, t)$ and $v_r = 0$, which satisfies continuity?
- 15 What is the most general form of a purely radial polar-coordinate incompressible-flow pattern, $v_r = v_r(r, \theta, t)$ and $v_\theta = 0$, which satisfies continuity?
- 16 An incompressible steady-flow pattern is given by $u = x^3 + 2z^2$ and $w = y^3 - 2yz$. What is the most general form of the third component, $v(x, y, z)$, which satisfies continuity?
- 17 A reasonable approximation for the two-dimensional incompressible laminar boundary layer on the flat surface in Fig. P 17 is

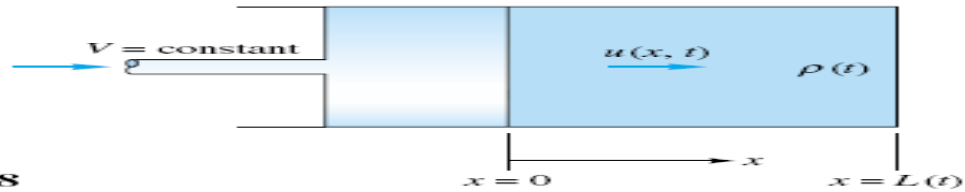
$$u = U \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \quad \text{for } y \leq \delta \quad \text{where } \delta = Cx^{1/2}, C = \text{const}$$

(a) Assuming a no-slip condition at the wall, find an expression for the velocity component $v(x, y)$ for $y \leq \delta$. (b) Then find the maximum value of v at the station $x = 1$ m, for the particular case of airflow, when $U = 3$ m/s and $\delta = 1.1$ cm.



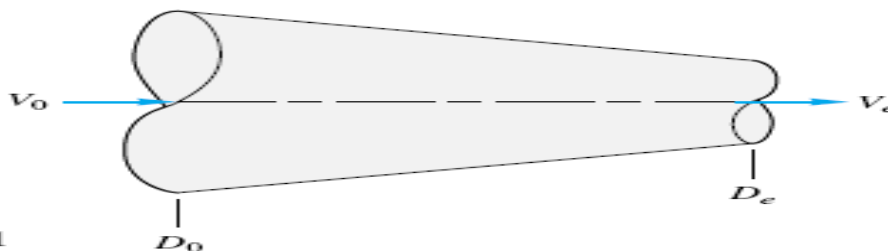
P .17

- 18 A piston compresses gas in a cylinder by moving at constant speed V , as in Fig. P .18. Let the gas density and length at $t = 0$ be ρ_0 and L_0 , respectively. Let the gas velocity vary linearly from $u = V$ at the piston face to $u = 0$ at $x = L$. If the gas density varies only with time, find an expression for $\rho(t)$.



P .18

- 19 An incompressible flow field has the cylindrical components $v_\theta = Cr$, $v_z = K(R^2 - r^2)$, $v_r = 0$, where C and K are constants and $r \leq R$, $z \leq L$. Does this flow satisfy continuity? What might it represent physically?
- 20 A two-dimensional incompressible velocity field has $u = K(1 - e^{-ay})$, for $x \leq L$ and $0 \leq y \leq \infty$. What is the most general form of $v(x, y)$ for which continuity is satisfied and $v = v_0$ at $y = 0$? What are the proper dimensions for constants K and a ?
- 21 Air flows under steady, approximately one-dimensional conditions through the conical nozzle in Fig. P .21. If the speed of sound is approximately 340 m/s, what is the minimum nozzle-diameter ratio D_e/D_0 for which we can safely neglect compressibility effects if $V_0 = (a)$ 10 m/s and (b) 30 m/s?



P .21

- 22 Air at a certain temperature and pressure flows through a contracting nozzle of length L whose area decreases linearly, $A \approx A_0[1 - x/(2L)]$. The air average velocity increases nearly linearly from 76 m/s at $x = 0$ to 167 m/s at $x = L$. If the density at $x = 0$ is 2.0 kg/m^3 , estimate the density at $x = L$.
- 23 A tank volume \mathcal{V} contains gas at conditions (ρ_0, p_0, T_0) . At time $t = 0$ it is punctured by a small hole of area A . According to the theory of Chap. 9, the mass flow out of such a hole is approximately proportional to A and to the tank pressure. If the tank temperature is assumed constant and the gas is ideal, find an expression for the variation of density within the tank.
- 24 Reconsider Fig. P .17 in the following general way. It is known that the boundary layer thickness $\delta(x)$ increases monotonically and that there is no slip at the wall ($y = 0$). Further, $u(x, y)$ merges smoothly with the outer stream flow, where $u \approx U = \text{constant}$ outside the layer. Use these facts to prove that (a) the component $v(x, y)$ is positive everywhere within the layer, (b) v increases parabolically with y very near the wall, and (c) v is a maximum at $y = \delta$.
- 25 An incompressible flow in polar coordinates is given by

$$v_r = K \cos \theta \left(1 - \frac{b}{r^2}\right)$$

$$v_\theta = -K \sin \theta \left(1 + \frac{b}{r^2}\right)$$

Does this field satisfy continuity? For consistency, what

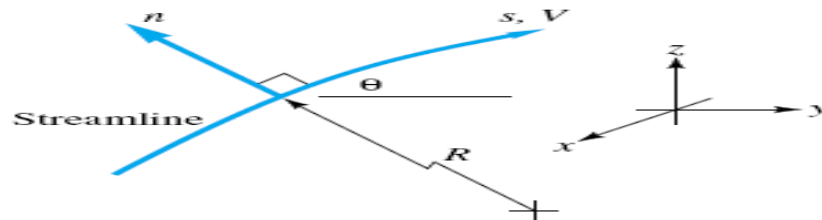
should the dimensions of constants K and b be? Sketch the surface where $v_r = 0$ and interpret.

- 26** Curvilinear, or streamline, coordinates are defined in Fig. P .26, where n is normal to the streamline in the plane of the radius of curvature R . Show that Euler's frictionless momentum equation () in streamline coordinates becomes

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial s} = -\frac{1}{\rho} \frac{\partial p}{\partial s} + g_s \quad (1)$$

$$-V \frac{\partial \theta}{\partial t} - \frac{V^2}{R} = -\frac{1}{\rho} \frac{\partial p}{\partial n} + g_n \quad (2)$$

Further show that the integral of Eq. (1) with respect to s is none other than our old friend Bernoulli's equation ().



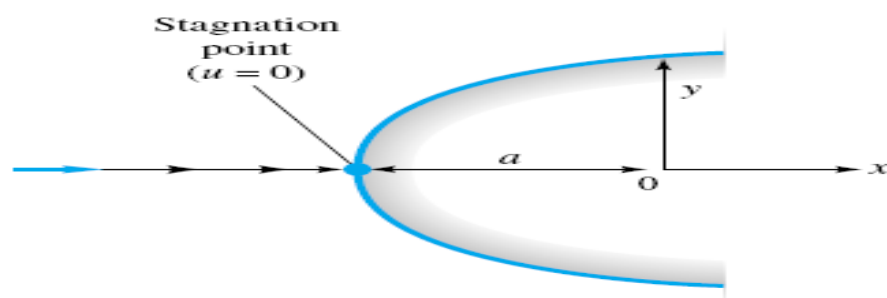
P .26

- 27** A frictionless, incompressible steady-flow field is given by

$$\mathbf{V} = 2xy\mathbf{i} - y^2\mathbf{j}$$

in arbitrary units. Let the density be $\rho_0 = \text{constant}$ and neglect gravity. Find an expression for the pressure gradient in the x direction.

- 28** If z is "up," what are the conditions on constants a and b for which the velocity field $u = ay$, $v = bx$, $w = 0$ is an exact solution to the continuity and Navier-Stokes equations for incompressible flow?
- 29** Consider a steady, two-dimensional, incompressible flow of a newtonian fluid in which the velocity field is known, i.e., $u = -2xy$, $v = y^2 - x^2$, $w = 0$. (a) Does this flow satisfy conservation of mass? (b) Find the pressure field, $p(x, y)$ if the pressure at the point $(x = 0, y = 0)$ is equal to p_a .
- 30** Show that the two-dimensional flow field of Example 1.10 is an exact solution to the incompressible Navier-Stokes equations (). Neglecting gravity, compute the pressure field $p(x, y)$ and relate it to the absolute velocity $V^2 = u^2 + v^2$. Interpret the result.
- 31** According to potential theory (Chap. 8) for the flow approaching a rounded two-dimensional body, as in Fig. P4.31, the velocity approaching the stagnation point is given by $u = U(1 - a^2/x^2)$, where a is the nose radius and U is the velocity far upstream. Compute the value and position of the maximum viscous normal stress along this streamline.

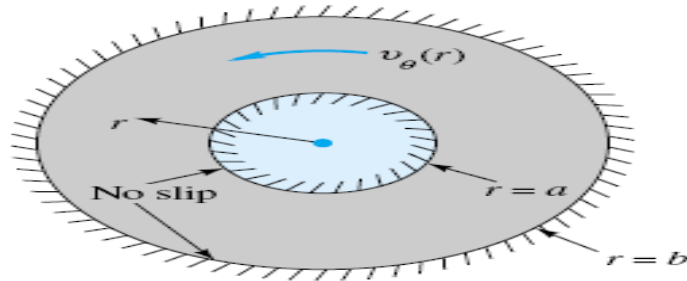


P 31

Is this also the position of maximum fluid deceleration? Evaluate the maximum viscous normal stress if the fluid is SAE 30 oil at 20°C , with $U = 2$ m/s and $a = 6$ cm.

- 32** The answer to Prob. .14 is $v_\theta = f(r)$ only. Do not reveal this to your friends if they are still working on Prob. 4.14. Show that this flow field is an exact solution to the Navier-Stokes equations () for only two special cases of the function $f(r)$. Neglect gravity. Interpret these two cases physically.
- 33** From Prob. .15 the purely radial polar-coordinate flow which satisfies continuity is $v_r = f(\theta)/r$, where f is an arbitrary function. Determine what particular forms of $f(\theta)$ satisfy the full Navier-Stokes equations in polar-coordinate form from Eqs. (D.5) and (D.6).

- 34 The fully developed laminar-pipe-flow solution of Prob. , $v_z = u_{\max}(1 - r^2/R^2)$, $v_\theta = 0$, $v_r = 0$, is an exact solution to the cylindrical Navier-Stokes equations (App. D). Neglecting gravity, compute the pressure distribution in the pipe $p(r, z)$ and the shear-stress distribution $\tau(r, z)$, using R , u_{\max} , and μ as parameters. Why does the maximum shear occur at the wall? Why does the density not appear as a parameter?
- 35 From the Navier-Stokes equations for incompressible flow in polar coordinates (App. D for cylindrical coordinates), find the most general case of purely circulating motion $v_\theta(r)$, $v_r = v_z = 0$, for flow with no slip between two fixed concentric cylinders, as in Fig. P .35.

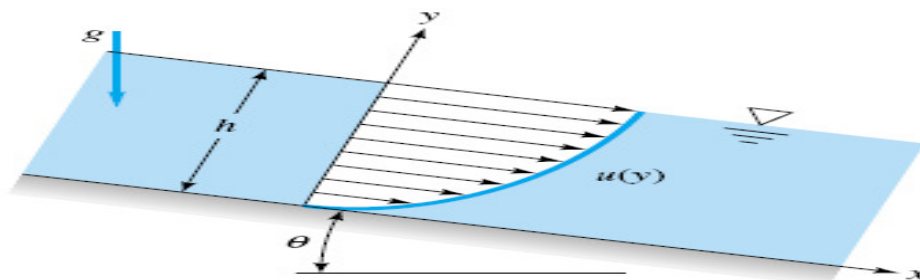


P .35

- 36 A constant-thickness film of viscous liquid flows in laminar motion down a plate inclined at angle θ , as in Fig. P .36. The velocity profile is

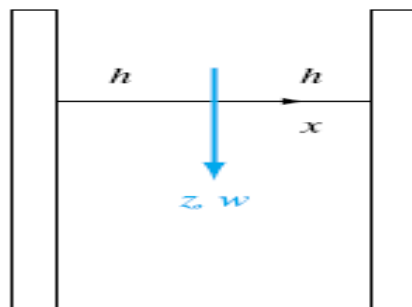
$$u = Cy(2h - y) \quad v = w = 0$$

Find the constant C in terms of the specific weight and viscosity and the angle θ . Find the volume flux Q per unit width in terms of these parameters.



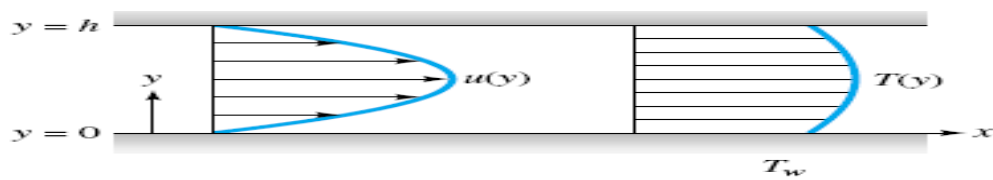
P .36

- 37 A viscous liquid of constant ρ and μ falls due to gravity between two plates a distance $2h$ apart, as in Fig. P4.37. The flow is fully developed, with a single velocity component $w = w(x)$. There are no applied pressure gradients, only gravity. Solve the Navier-Stokes equation for the velocity profile between the plates.



P .37

- 38 Reconsider the angular-momentum balance of Fig. by adding a concentrated *body couple* C_z about the z axis [6]. Determine a relation between the body couple and shear stress for equilibrium. What are the proper dimensions for C_z ? (Body couples are important in continuous media with microstructure, such as granular materials.)
- 39 Problems involving viscous dissipation of energy are dependent on viscosity μ , thermal conductivity k , stream velocity U_0 , and stream temperature T_0 . Group these parameters into the dimensionless *Brinkman number*, which is proportional to μ .
- 40 As mentioned in Sec. , the velocity profile for laminar flow between two plates, as in Fig. P 40, is



P 40

$$u = \frac{4u_{\max}y(h-y)}{h^2} \quad v = w = 0$$

If the wall temperature is T_w at both walls, use the incompressible-flow energy equation () to solve for the temperature distribution $T(y)$ between the walls for steady flow.

- 41** The approximate velocity profile in Prob. and Fig. for steady laminar flow through a duct, was suggested as

$$u = u_{\max} \left(1 - \frac{y^2}{b^2}\right) \left(1 - \frac{z^2}{h^2}\right)$$

With $v = w = 0$, it satisfied the no-slip condition and gave a reasonable volume-flow estimate (which was the point of Prob.). Show, however, that it does *not* satisfy the x -momentum Navier-Stokes equation for duct flow with constant pressure gradient $\partial p/\partial x < 0$. For extra credit, explain briefly how the actual exact solution to this problem is obtained (see, for example, Ref. 5, pp. 120–121).

- 42** In duct-flow problems with heat transfer, one often defines an average fluid temperature. Consider the duct flow of Fig. P 40 of width b into the paper. Using a control-volume integral analysis with constant density and specific heat, derive an expression for the temperature arising if the entire duct flow poured into a bucket and was stirred uniformly. Assume arbitrary $u(y)$ and $T(y)$. This average is called the *cup-mixing temperature* of the flow.
- 43** For the draining liquid film of Fig. P 36, what are the appropriate boundary conditions (a) at the bottom $y = 0$ and (b) at the surface $y = h$?
- 44** Suppose that we wish to analyze the sudden pipe-expansion flow of Fig. , using the full continuity and Navier-Stokes equations. What are the proper boundary conditions to handle this problem?
- 45** Suppose that we wish to analyze the U-tube oscillation flow of Fig. , using the full continuity and Navier-Stokes equations. What are the proper boundary conditions to handle this problem?

References

1. D. A. Anderson, J. C. Tannehill, and R. H. Pletcher, *Computational Fluid Mechanics and Heat Transfer*, 2d ed., McGraw-Hill, New York, 1997.
2. G. B. Wallis, *One-Dimensional Two-Phase Flow*, McGraw-Hill, New York, 1969.
3. S. M. Selby, *CRC Handbook of Tables for Mathematics*, 4th ed., CRC Press Inc., Cleveland, OH, 1976.
4. H. Schlichting, *Boundary Layer Theory*, 7th ed., McGraw-Hill, New York, 1979.
5. F. M. White, *Viscous Fluid Flow*, 2d ed., McGraw-Hill, New York, 1991.
6. L. E. Malvern, *Introduction to Mechanics of a Continuous Medium*, Prentice-Hall, Englewood Cliffs, NJ, 1969.
7. J. P. Holman, *Heat Transfer*, 8th ed., McGraw-Hill, New York, 1997.
8. W. M. Kays and M. E. Crawford, *Convective Heat and Mass Transfer*, 3d ed., McGraw-Hill, New York, 1993.
9. G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, England, 1967.
10. L. Prandtl and O. G. Tietjens, *Fundamentals of Hydro- and Aeromechanics*, Dover, New York, 1957.
11. R. Aris, *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*, Prentice-Hall, Englewood Cliffs, NJ, 1962.
12. G. E. Mase, *Continuum Mechanics*, Schaum's Outline Series, McGraw-Hill, New York, 1970.
13. W. F. Hughes and E. W. Gaylord, *Basic Equations of Engineering Science*, Schaum's Outline Series, McGraw-Hill, New York, 1964.
14. G. Astarita and G. Marrucci, *Principles of Non-Newtonian Fluid Mechanics*, McGraw-Hill, New York, 1974.
15. H. Lamb, *Hydrodynamics*, 6th ed., Dover, New York, 1945.
16. A. Szeri, *Tribology: Friction, Lubrication, and Wear*, McGraw-Hill, New York, 1980.
17. G. I. Taylor, "Stability of a Viscous Liquid Contained between Two Rotating Cylinders," *Philos. Trans. Roy. Soc. London Ser. A*, vol. 223, 1923, pp. 289–343.
18. E. L. Koschmieder, "Turbulent Taylor Vortex Flow," *J. Fluid Mech.*, vol. 93, pt. 3, 1979, pp. 515–527.

End of Part-1