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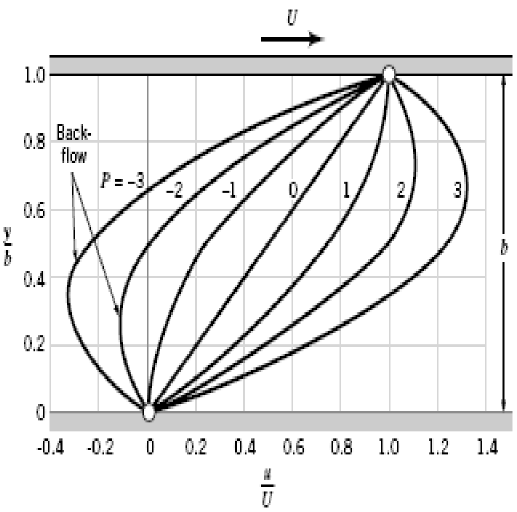

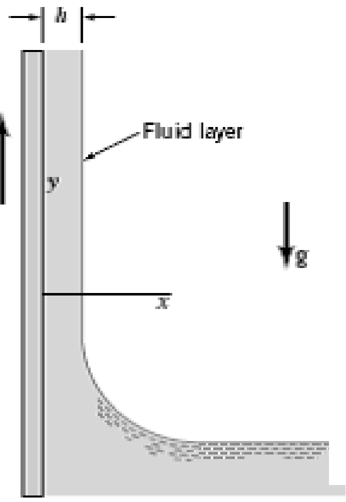

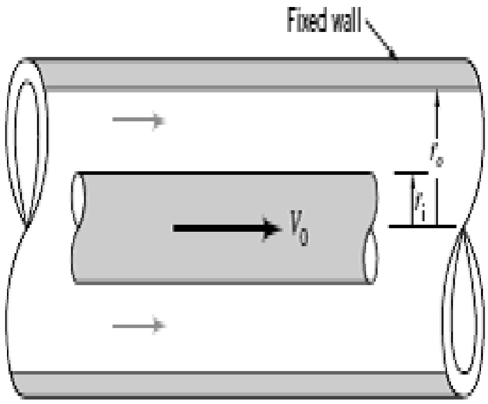

Notes on the course

Fluid Mechanics (3) - MEP 303A

For THRID YEAR MECHANICS (POWER)

Part (2)

Examples on Differential Analysis of Incompressible Viscous Flow

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Examples on Differential Analysis of Incompressible Viscous Flow

Summary of Conservation Differential Equations (from Part 1):

(I) Conservation of Mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad (2.1)$$

As previously mentioned, this equation is also commonly referred to as the continuity equation.

The continuity equation is one of the fundamental equations of fluid mechanics and, as expressed in Eq. 2.1 , is valid for steady or unsteady flow, and compressible or incompressible fluids. In vector notation, Eq. 2.1 can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0 \qquad (2.2)$$

Two special cases are of particular interest. For *steady* flow of *compressible* fluids

$$\nabla \cdot \rho \mathbf{V} = 0$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad (2.3)$$

This follows since by definition ρ is not a function of time for steady flow, but could be a function of position. For *incompressible* fluids the fluid density, ρ, is a constant throughout the flow field so that Eq. 2.2 becomes

$$\nabla \cdot \mathbf{V} = 0 \qquad (2.4)$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (2.5)$$

Equation 2.5 applies to both steady and unsteady flow of incompressible fluids. Note that Eq. 2.5 is the same as that obtained by setting the volumetric dilatation rate (Eq. 1.9) equal to zero. This result should not be surprising since both relationships are based on conservation of mass for incompressible fluids. However, the expression for the volumetric dilation rate was developed from a system approach, whereas Eq. 2.5 was developed from a control volume approach. In the former case the deformation of a particular differential mass of fluid was studied, and in the latter case mass flow through a fixed differential volume was studied.

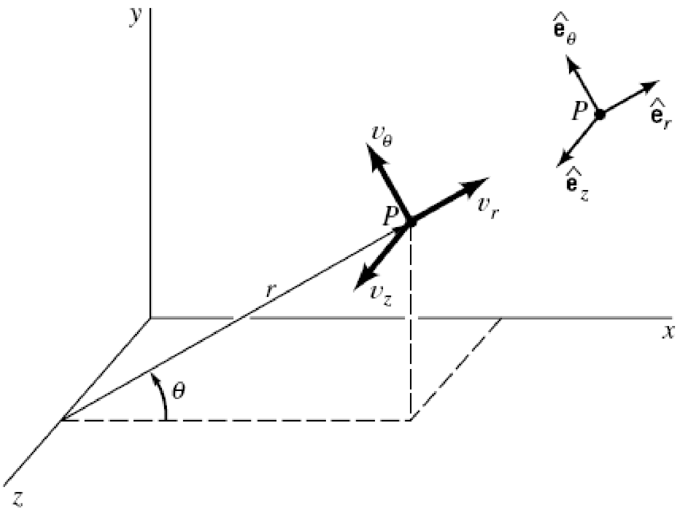
* Ref.:(1) Bruce R. Munson, Donald F. Young, Theodore H. Okiishi "Fundamental of Fluid Mechanics" 4th ed., John Wiley & Sons, Inc., 2002.
(2) Frank M. White "Fluid Mechanics", 4th ed. McGraw Hill, 2002.

The differential form of the continuity equation in cylindrical coordinates is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0 \qquad (2.6)$$

This equation can be derived by following the same procedure used in the preceding section For steady, compressible flow

$$\frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0 \qquad (2.7)$$



■ FIGURE 2.1 **The representation of velocity components in cylindrical polar coordinates.**

For incompressible fluids (for steady or unsteady flow)

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \qquad (2.8)$$

(II) Conservation of Linear Momentum:

The Navier–Stokes Equations

(x direction):

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad (2.9a)$$

(y direction)

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad (2.9b)$$

(z direction)

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \qquad (2.9c)$$

In terms of cylindrical polar coordinates (see Fig. 1.6), the Navier–Stokes equation can be written as

(r direction)

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial r} + \rho g_r + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] \quad (2.10a)$$

(θ direction)

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] \quad (2.10b)$$

(z direction)

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] \quad (2.10c)$$

2.1 Some Simple Solutions for Viscous, Incompressible Fluids

A principal difficulty in solving the Navier–Stokes equations is because of their nonlinearity arising from the convective acceleration terms (i.e., $u\,\partial u/\partial x$, $w\,\partial v/\partial z$, etc.). There are no general analytical schemes for solving nonlinear partial differential equations (e.g., superposition of solutions cannot be used), and each problem must be considered individually. For most practical flow problems, fluid particles do have accelerated motion as they move from one location to another in the flow field. Thus, the convective acceleration terms are usually important. However, there are a few special cases for which the convective acceleration vanishes because of the nature of the geometry of the flow system. In these cases exact solutions are usually possible. The Navier–Stokes equations apply to both laminar and turbulent flow, but for turbulent flow each velocity component fluctuates randomly with respect to time and this added complication makes an analytical solution intractable. Thus, the exact solutions referred to are for laminar flows in which the velocity is either independent of time (steady flow) or dependent on time (unsteady flow) in a well-defined manner.

2.1.1 Steady, Laminar Flow Between Fixed Parallel Plates

We first consider flow between the two horizontal, infinite parallel plates of Fig. 2.2 a . For this geometry the fluid particles move in the x direction parallel to the plates, and there is no velocity in the y or z direction—that is, $v = 0$ and $w = 0$. In this case it follows from the continuity equation (Eq. 2.5) that $\partial u/\partial x = 0$. Furthermore, there would be no variation of u in the z direction for infinite plates, and for steady flow $\partial u/\partial t = 0$ so that $u = u(y)$. If these conditions are used in the Navier–Stokes equations (Eqs. 2.9), they reduce to

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right) \qquad (2.11)$$

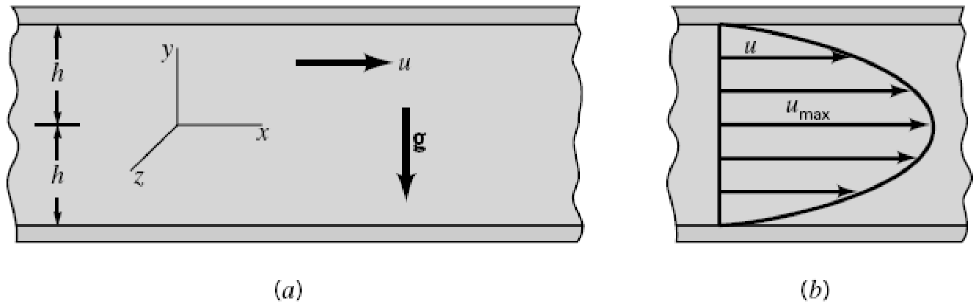
$$0 = -\frac{\partial p}{\partial y} - \rho g \qquad (2.12)$$

$$0 = -\frac{\partial p}{\partial z} \qquad (2.13)$$

where we have set $g_x = 0$, $g_y = -g$, and $g_z = 0$. That is, the y axis points up. We see that for this particular problem the Navier–Stokes equations reduce to some rather simple equations.

Equations 2.12 and 2.13 can be integrated to yield

$$p = -\rho g y + f_1(x) \qquad (2.14)$$



■ FIGURE 2.2 **The viscous flow between parallel plates: (*a*) coordinate system and notation used in analysis; (*b*) parabolic velocity distribution for flow between parallel fixed plates.**

which shows that the pressure varies hydrostatically in the y direction. Equation 2.11 , rewritten as

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

can be integrated to give

$$\frac{du}{dy} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)y + c_1$$

and integrated again to yield

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + c_1 y + c_2 \qquad (2.15)$$

Note that for this simple flow the pressure gradient, $\partial p/\partial x$, is treated as constant as far as the integration is concerned, since (as shown in Eq. 2.14) it is not a function of y. The two constants c_1 and c_2 must be determined from the boundary conditions. For example, if the two plates are fixed, then $u = 0$ for $y = \pm h$ (because of the no-slip condition for viscous fluids). To satisfy this condition $c_1 = 0$ and

$$c_2 = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)h^2$$

Thus, the velocity distribution becomes

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h^2) \tag{2.16}$$

Equation 2.16 shows that the velocity profile between the two fixed plates is parabolic as illustrated in Fig. 2.2 b.

The volume rate of flow, q, passing between the plates (for a unit width in the z direction) is obtained from the relationship

$$q = \int_{-h}^{h} u\, dy = \int_{-h}^{h} \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h^2)\, dy$$

or

$$q = -\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right) \tag{2.17}$$

The pressure gradient $\partial p/\partial x$ is negative, since the pressure decreases in the direction of flow. If we let Δp represent the pressure *drop* between two points a distance ℓ apart, then

$$\frac{\Delta p}{\ell} = -\frac{\partial p}{\partial x}$$

and Eq. 2.17 can be expressed as

$$q = \frac{2h^3 \Delta p}{3\mu\ell} \tag{2.18}$$

The flow is proportional to the pressure gradient, inversely proportional to the viscosity, and strongly dependent ($\sim h^3$) on the gap width. In terms of the mean velocity, V, where $V = q/2h$, Eq. 2.18 becomes

$$V = \frac{h^2 \Delta p}{3\mu\ell} \tag{2.19}$$

Equations 2.18 and 2.19 provide convenient relationships for relating the pressure drop along a parallel-plate channel and the rate of flow or mean velocity. The maximum velocity, u_{max}, occurs midway ($y = 0$) between the two plates so that from Eq. 2.16

$$u_{max} = -\frac{h^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)$$

or

$$u_{max} = \tfrac{3}{2}V \tag{2.20}$$

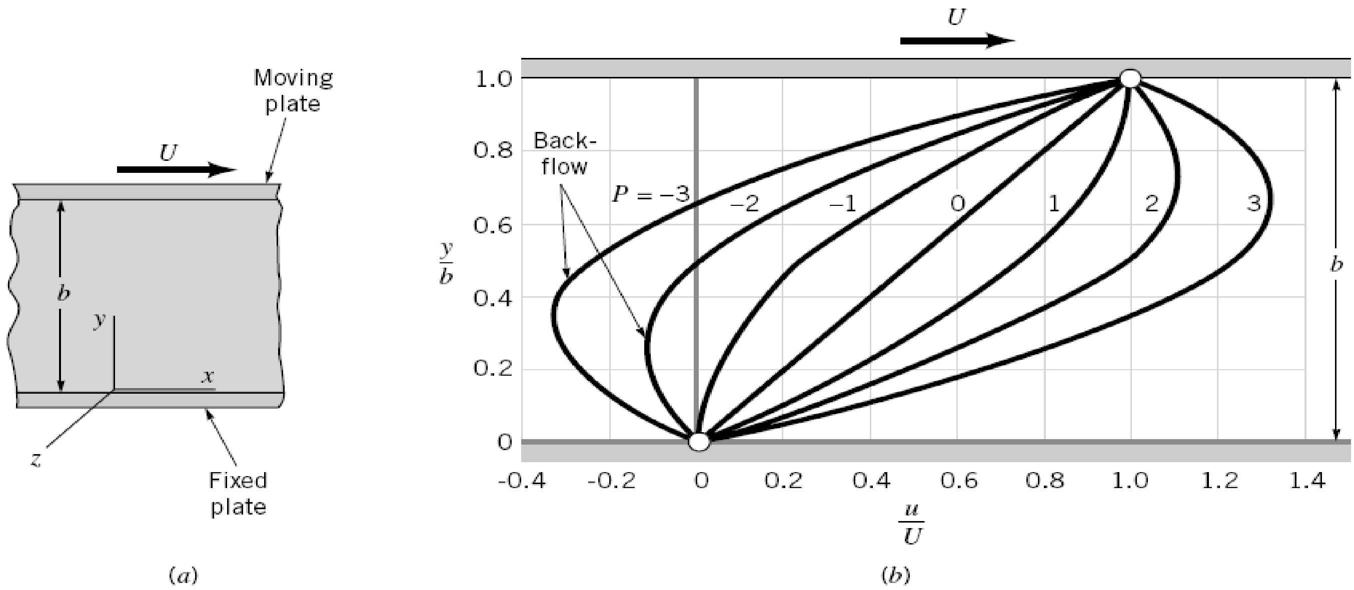
The details of the steady laminar flow between infinite parallel plates are completely predicted by this solution to the Navier–Stokes equations. For example, if the pressure gradient, viscosity, and plate spacing are specified, then from Eq. 2.16 the velocity profile can be determined, and from Eqs. 2.18 and 2.19 the corresponding flowrate and mean velocity determined. In addition, from Eq. 2.14 it follows that

$$f_1(x) = \left(\frac{\partial p}{\partial x}\right)x + p_0$$

where p_0 is a reference pressure at $x = y = 0$, and the pressure variation throughout the fluid can be obtained from

$$p = -\rho g y + \left(\frac{\partial p}{\partial x}\right) x + p_0 \qquad (2.21)$$

For a given fluid and reference pressure, p_0, the pressure at any point can be predicted. This relatively simple example of an exact solution illustrates the detailed information about the flow field which can be obtained. The flow will be laminar if the Reynolds number, $\mathrm{Re} = \rho V(2h)/\mu$, remains below about 1400. For flow with larger Reynolds numbers the flow becomes turbulent and the preceding analysis is not valid since the flow field is complex, three-dimensional, and unsteady.



■ FIGURE 2.3 **The viscous flow between parallel plates with bottom plate fixed and upper plate moving (Couette flow): (*a*) coordinate system and notation used in analysis; (*b*) velocity distribution as a function of parameter, *P*, where $P = -(b^2/2\mu U)\,\partial p/\partial x$. (From Ref. 8, used by permission.)**

2.1.2 Couette Flow

Another simple parallel-plate flow can be developed by fixing one plate and letting the other plate move with a constant velocity, U, as is illustrated in Fig. 2.3 a . The Navier–Stokes equations reduce to the same form as those in the preceding section, and the solution for the pressure and velocity distribution are still given by Eqs. 2.14 and 2.15 , respectively. However, for the moving plate problem the boundary conditions for the velocity are different. For this case we locate the origin of the coordinate system at the bottom plate and designate the distance between the two plates as b (see Fig. 2.3 a). The two constants c_1 and c_2 in Eq. 2.15 can be determined from the boundary conditions, $u = 0$ at $y = 0$ and $u = U$ at $y = b$. It follows that

$$u = U\frac{y}{b} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - by) \qquad (2.22)$$

or, in dimensionless form,

$$\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)\left(\frac{y}{b}\right)\left(1 - \frac{y}{b}\right) \qquad (2.23)$$

The actual velocity profile will depend on the dimensionless parameter

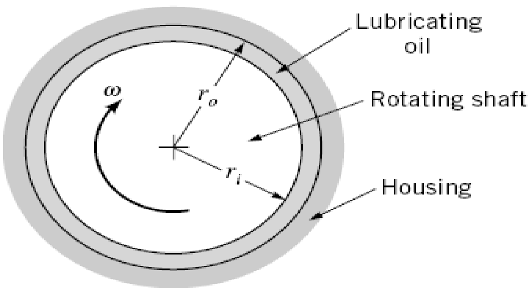
$$P = -\frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)$$

Several profiles are shown in Fig. 2.3 b . This type of flow is called *Couette flow*.

The simplest type of Couette flow is one for which the pressure gradient is zero; that is, the fluid motion is caused by the fluid being dragged along by the moving boundary. In this case, with $\partial p/\partial x = 0$, Eq. 2.22 simply reduces to

$$u = U\frac{y}{b} \quad (2.24)$$

which indicates that the velocity varies linearly between the two plates as shown in Fig. 2.3 b for $P = 0$. This situation would be approximated by the flow between closely spaced concentric cylinders in which one cylinder is fixed and the other cylinder rotates with a constant angular velocity, ω. As illustrated in Fig. 2.4 , the flow in an unloaded journal bearing might be approximated by this simple Couette flow if the gap width is very small (i.e., $r_o - r_i \ll r_i$). In this case $U = r_i\omega$, $b = r_o - r_i$, and the shearing stress resisting the rotation of the shaft can be simply calculated as $\tau = \mu r_i \omega/(r_o - r_i)$. When the bearing is loaded (i.e., a force applied normal to the axis of rotation) the shaft will no longer remain concentric with the housing and the flow cannot be treated as flow between parallel boundaries. Such problems are dealt with in lubrication theory (see, for example, Ref. 9).



■ FIGURE 2.4 **Flow in the narrow gap of a journal bearing.**

Couette Flow Between a Fixed and a Moving Plate:

Consider two-dimensional incompressible plane ($\partial/\partial z = 0$) viscous flow between parallel plates a distance $2h$ apart, as shown in Fig. 2.5 . We assume that the plates are very wide and very long, so that the flow is essentially axial, $u \neq 0$ but $v = w = 0$. The present case is Fig. 2.5a , where the upper plate moves at velocity V but there is no pressure gradient. Neglect gravity effects. We learn from the continuity equation that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial u}{\partial x} + 0 + 0 \qquad \text{or} \qquad u = u(y) \text{ only}$$

Thus there is a single nonzero axial-velocity component which varies only across the channel. The flow is said to be *fully developed* (far downstream of the entrance). Substitute $u = u(y)$ into the x-component of the Navier-Stokes momentum equation for two-dimensional (x, y) flow:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

or

$$\rho(0 + 0) = 0 + 0 + \mu\left(0 + \frac{d^2 u}{dy^2}\right) \quad (2.25)$$

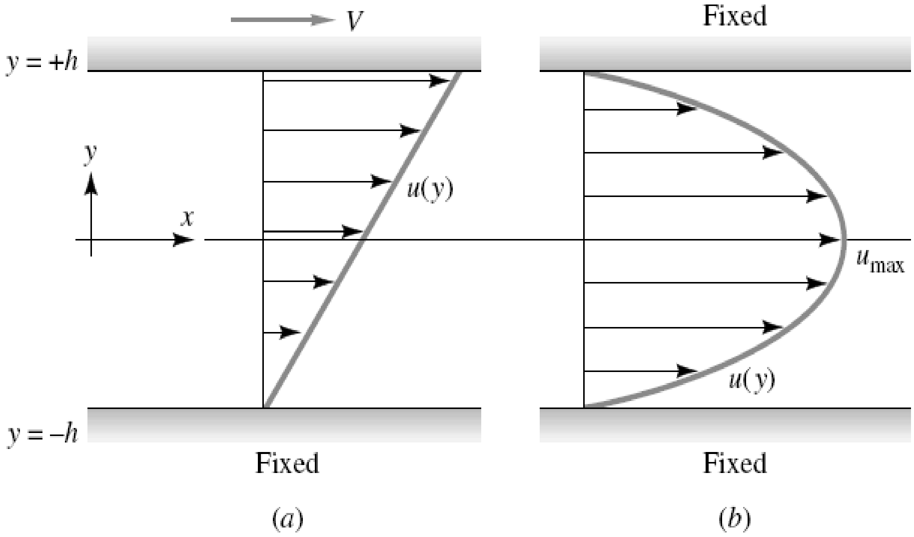


Fig. 2.5 Incompressible viscous flow between parallel plates: (*a*) no pressure gradient, upper plate moving; (*b*) pressure gradient $\partial p/\partial x$ with both plates fixed.

Most of the terms drop out, and the momentum equation simply reduces to

$$\frac{d^2u}{dy^2} = 0 \qquad \text{or} \qquad u = C_1 y + C_2$$

The two constants are found by applying the no-slip condition at the upper and lower plates:

At $y = +h$: $\qquad u = V = C_1 h + C_2$

At $y = -h$: $\qquad u = 0 = C_1(-h) + C_2$

or $\qquad C_1 = \frac{V}{2h} \qquad \text{and} \qquad C_2 = \frac{V}{2}$

Therefore the solution for this case (*a*), flow between plates with a moving upper wall, is

$$u = \frac{V}{2h} y + \frac{V}{2} \qquad -h \le y \le +h \qquad (2.26)$$

This is *Couette flow* due to a moving wall: a linear velocity profile with no-slip at each wall, as anticipated and sketched in Fig. 2.5a . Note that the origin has been placed in the center of the channel, for convenience in case (*b*) below.

Flow Due to Pressure Gradient Between Two Fixed Plates:

Case (*b*) is sketched in Fig. 2.5b . Both plates are fixed ($V = 0$), but the pressure varies in the *x* direction. If $v = w = 0$, the continuity equation leads to the same conclusion as case (*a*), namely, that $u = u(y)$ only. The *x*-momentum equation 2.127a changes only because the pressure is variable:

$$\mu \frac{d^2u}{dy^2} = \frac{\partial p}{\partial x} \qquad (2.27)$$

Also, since $v = w = 0$ and gravity is neglected, the *y*- and *z*-momentum equations lead to

$$\frac{\partial p}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial p}{\partial z} = 0 \qquad \text{or} \qquad p = p(x) \text{ only}$$

Thus the pressure gradient in Eq. (2.27) is the total and only gradient:

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} = \text{const} < 0 \tag{2.28}$$

Why did we add the fact that *dp/dx* is *constant*? Recall a useful conclusion from the theory of separation of variables: If two quantities are equal and one varies only with *y* and the other varies only with *x*, then they must both equal the same constant. Otherwise they would not be independent of each other.

Why did we state that the constant is *negative*? Physically, the pressure must decrease in the flow direction in order to drive the flow against resisting wall shear stress. Thus the velocity profile *u*(*y*) must have negative curvature everywhere, as anticipated and sketched in Fig. 2.5b.

The solution to Eq. (2.28) is accomplished by double integration:

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C_1 y + C_2$$

The constants are found from the no-slip condition at each wall:

$$\text{At } y = \pm h: \qquad u = 0 \qquad \text{or} \qquad C_1 = 0 \qquad \text{and} \qquad C_2 = -\frac{dp}{dx}\frac{h^2}{2\mu}$$

Thus the solution to case (*b*), flow in a channel due to pressure gradient, is

$$u = -\frac{dp}{dx}\frac{h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right) \tag{2.29}$$

The flow forms a *Poiseuille* parabola of constant negative curvature. The maximum velocity occurs at the centerline $y = 0$:

$$u_{max} = -\frac{dp}{dx}\frac{h^2}{2\mu} \tag{2.30}$$

Other (laminar) flow parameters are computed in the following example.

EXAMPLE 2.1

For case (*b*) above, flow between parallel plates due to the pressure gradient, compute (*a*) the wall shear stress, (*b*) the stream function, (*c*) the vorticity, (*d*) the velocity potential, and (*e*) the average velocity.

Solution

All parameters can be computed from the basic solution, Eq. (2.29), by mathematical manipulation.

(*a*) The wall shear follows from the definition of a newtonian fluid

$$\tau_w = \tau_{xy\,\text{wall}} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\bigg|_{y=\pm h} = \mu\frac{\partial}{\partial y}\left[\left(-\frac{dp}{dx}\right)\left(\frac{h^2}{2\mu}\right)\left(1 - \frac{y^2}{h^2}\right)\right]\bigg|_{y=\pm h}$$

$$= \pm\frac{dp}{dx}h = \mp\frac{2\mu u_{max}}{h} \qquad \textit{Ans. (a)}$$

The wall shear has the same magnitude at each wall, but by our sign convention of the upper wall has negative shear stress.

(*b*) Since the flow is plane, steady, and incompressible, a stream function exists:

$$u = \frac{\partial\psi}{\partial y} = u_{max}\left(1 - \frac{y^2}{h^2}\right) \qquad v = -\frac{\partial\psi}{\partial x} = 0$$

Integrating and setting $\psi = 0$ at the centerline for convenience, we obtain

$$\psi = u_{max}\left(y - \frac{y^3}{3h^2}\right) \qquad \textit{Ans. (b)}$$

At the walls, $y = \pm h$ and $\psi = \pm 2u_{max}h/3$, respectively.

(*c*) In plane flow, there is only a single nonzero vorticity component:

$$\zeta_z = (\text{curl } \mathbf{V})_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{2u_{max}}{h^2}\, y \qquad \textit{Ans. (c)}$$

The vorticity is highest at the wall and is positive (counterclockwise) in the upper half and negative (clockwise) in the lower half of the fluid. Viscous flows are typically full of vorticity and are not at all irrotational.

(*d*) From part (*c*), the vorticity is finite. Therefore the flow is not irrotational, and the velocity potential *does not exist.* *Ans. (d)*

(*e*) The average velocity is defined as $V_{av} = Q/A$, where $Q = \int u\, dA$ over the cross section. For our particular distribution $u(y)$ from Eq. (2.29), we obtain

$$V_{av} = \frac{1}{A}\int u\, dA = \frac{1}{b(2h)}\int_{-h}^{+h} u_{max}\left(1 - \frac{y^2}{h^2}\right) b\, dy = \frac{2}{3}\, u_{max} \qquad \textit{Ans. (e)}$$

In plane Poiseuille flow between parallel plates, the average velocity is two-thirds of the maximum (or centerline) value. This result could also have been obtained from the stream function derived in part (*b*).

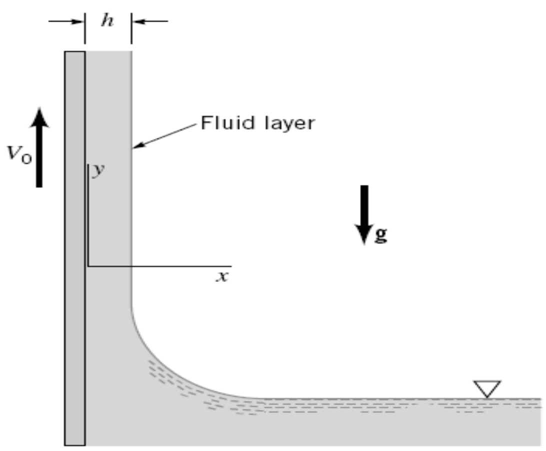
$$Q_{channel} = \psi_{upper} - \psi_{lower} = \frac{2u_{max}h}{3} - \left(-\frac{2u_{max}h}{3}\right) = \frac{4}{3}\, u_{max}h \text{ per unit width}$$

whence $V_{av} = Q/A_{b=1} = (4u_{max}h/3)/(2h) = 2u_{max}/3$, the same result.

This example illustrates a statement made earlier: Knowledge of the velocity vector **V** [as in Eq. (2.29)] is essentially the *solution* to a fluid-mechanics problem, since all other flow properties can then be calculated.

Example 2.2:

A wide moving belt passes through a container of a viscous liquid. The belt moves vertically upward with a constant velocity, V_0, as illustrated in Fig. E2.2. Because of viscous forces the belt picks up a film of fluid of thickness h. Gravity tends to make the fluid drain down the belt. Use the Navier–Stokes equations to determine an expression for the average velocity of the fluid film as it is dragged up the belt. Assume that the flow is laminar, steady, and fully developed.



■ FIGURE E2.2

SOLUTION

Since the flow is assumed to be fully developed, the only velocity component is in the y direction (the v component) so that $u = w = 0$. It follows from the continuity equation that $\partial v/\partial y = 0$, and for steady flow $\partial v/\partial t = 0$, so that $v = v(x)$. Under these conditions the Navier–Stokes equations for the x direction (Eq. 2.9 a) and the z direction (perpendicular to the paper) (Eq. 2.9 c) simply reduce to

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial z} = 0$$

This result indicates that the pressure does not vary over a horizontal plane, and since the pressure on the surface of the film ($x = h$) is atmospheric, the pressure throughout the film must be atmospheric (or zero gage pressure). The equation of motion in the y direction (Eq. 2.9 b) thus reduces to

$$0 = -\rho g + \mu \frac{d^2 v}{dx^2}$$

or

$$\frac{d^2 v}{dx^2} = \frac{\gamma}{\mu} \qquad \textbf{(1)}$$

Integration of Eq. 1 yields

$$\frac{dv}{dx} = \frac{\gamma}{\mu} x + c_1 \qquad \textbf{(2)}$$

On the film surface ($x = h$) we assume the shearing stress is zero—that is, the drag of the air on the film is negligible. The shearing stress at the free surface (or any interior parallel surface) is designated as τ_{xy},

$$\tau_{xy} = \mu \left(\frac{dv}{dx} \right)$$

Thus, if $\tau_{xy} = 0$ at $x = h$, it follows from Eq. 2 that

$$c_1 = -\frac{\gamma h}{\mu}$$

A second integration of Eq. 2 gives the velocity distribution in the film as

$$v = \frac{\gamma}{2\mu} x^2 - \frac{\gamma h}{\mu} x + c_2$$

At the belt ($x = 0$) the fluid velocity must match the belt velocity, V_0, so that

$$c_2 = V_0$$

and the velocity distribution is therefore

$$v = \frac{\gamma}{2\mu} x^2 - \frac{\gamma h}{\mu} x + V_0$$

With the velocity distribution known we can determine the flowrate per unit width, q, from the relationship

$$q = \int_0^h v \, dx = \int_0^h \left(\frac{\gamma}{2\mu} x^2 - \frac{\gamma h}{\mu} x + V_0 \right) dx$$

and thus

$$q = V_0 h - \frac{\gamma h^3}{3\mu}$$

The average film velocity, V (where $q = Vh$), is therefore

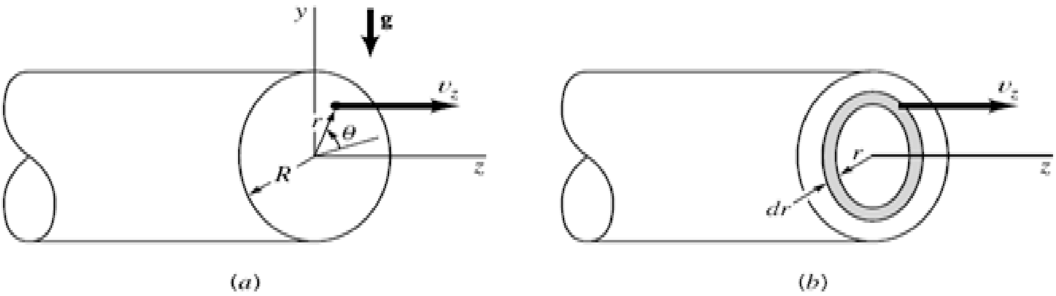
$$V = V_0 - \frac{\gamma h^2}{3\mu} \qquad \textbf{(Ans)}$$

It is interesting to note from this result that there will be a net upward flow of liquid (positive V) only if $V_0 > \gamma h^2 / 3\mu$. It takes a relatively large belt speed to lift a small viscosity fluid.

2.1.3 Steady, Laminar Flow in Circular Tubes

Probably the best known exact solution to the Navier–Stokes equations is for steady, incompressible, laminar flow through a straight circular tube of constant cross section. This type of flow is commonly called *Hagen-Poiseuille flow*, or simply *Poiseuille flow*. It is named in honor of J. L. Poiseuille (1799–1869), a French physician, and G. H. L. Hagen (1797–1884), a German hydraulic engineer. Poiseuille was interested in blood flow through capillaries and deduced experimentally the resistance laws for laminar flow through circular tubes. Hagen's investigation of flow in tubes was also experimental. It was actually after the work of Hagen and Poiseuille that the theoretical results presented in this section were determined, but their names are commonly associated with the solution of this problem.

Consider the flow through a horizontal circular tube of radius R as is shown in Fig. 2.6 a. Because of the cylindrical geometry it is convenient to use cylindrical coordinates. We assume that the flow is parallel to the walls so that $v_r = 0$ and $v_\theta = 0$, and from the continuity equation (2.7) $\partial v_z/\partial z = 0$. Also, for steady, axisymmetric flow, v_z is not a function of t or θ so the velocity, v_z, is only a function of the radial position within the tube—



■ FIGURE 2.6 **The viscous flow in a horizontal, circular tube: (*a*) coordinate system and notation used in analysis; (*b*) flow through differential annular ring.**

that is, $v_z = v_z(r)$. Under these conditions the Navier–Stokes equations (Eqs. 2.10) reduce to

$$0 = -\rho g \sin\theta - \frac{\partial p}{\partial r} \tag{2.31}$$

$$0 = -\rho g \cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \tag{2.32}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] \tag{2.33}$$

where we have used the relationships $g_r = -g\sin\theta$ and $g_\theta = -g\cos\theta$ (with θ measured from the horizontal plane).

Equations 2.31 and 2.32 can be integrated to give

$$p = -\rho g(r\sin\theta) + f_1(z)$$

or

$$p = -\rho g y + f_1(z) \tag{2.34}$$

Equation 2.34 indicates that the pressure is hydrostatically distributed at any particular cross section, and the z component of the pressure gradient, $\partial p/\partial z$, is not a function of r or θ.

The equation of motion in the z direction (Eq. 2.33) can be written in the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z}$$

and integrated (using the fact that $\partial p/\partial z$ = constant) to give

$$r\frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right)r^2 + c_1$$

Integrating again we obtain

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r^2 + c_1\ln r + c_2 \tag{2.35}$$

Since we wish v_z to be finite at the center of the tube ($r = 0$), it follows that $c_1 = 0$ [since $\ln(0) = -\infty$]. At the wall ($r = R$) the velocity must be zero so that

$$c_2 = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)R^2$$

and the velocity distribution becomes

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r^2 - R^2) \qquad (2.36)$$

Thus, at any cross section the velocity distribution is parabolic.

To obtain a relationship between the volume rate of flow, Q, passing through the tube and the pressure gradient, we consider the flow through the differential, washer-shaped ring of Fig. 2.6 b . Since v_z is constant on this ring, the volume rate of flow through the differential area $dA = (2\pi r)\, dr$ is

$$dQ = v_z(2\pi r)\, dr$$

and therefore

$$Q = 2\pi \int_0^R v_z r\, dr \qquad (2.37)$$

Equation 2.36 for v_z can be substituted into Eq. 2.37 , and the resulting equation integrated to yield

$$Q = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right) \qquad (2.38)$$

This relationship can be expressed in terms of the pressure *drop*, Δp, which occurs over a length, ℓ, along the tube, since

$$\frac{\Delta p}{\ell} = -\frac{\partial p}{\partial z}$$

and therefore

$$Q = \frac{\pi R^4 \Delta p}{8\mu\ell} \qquad (2.39)$$

For a given pressure drop per unit length, the volume rate of flow is inversely proportional to the viscosity and proportional to the tube radius to the fourth power. A doubling of the tube radius produces a sixteenfold increase in flow! Equation 2.39 is commonly called *Poiseuille's law.*

In terms of the mean velocity, V, where $V = Q/\pi R^2$, Eq. 2.39 becomes

$$V = \frac{R^2 \Delta p}{8\mu\ell} \qquad (2.40)$$

The maximum velocity v_{max} occurs at the center of the tube, where from Eq. 2.36

$$v_{max} = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{R^2 \Delta p}{4\mu\ell} \qquad (2.41)$$

so that

$$v_{max} = 2V$$

The velocity distribution can be written in terms of v_{max} as (2.42)

$$\frac{v_z}{v_{max}} = 1 - \left(\frac{r}{R}\right)^2 \qquad \textbf{(6.154)}$$

As was true for the similar case of flow between parallel plates (sometimes referred to as *plane Poiseuille flow*), a very detailed description of the pressure and velocity distribution in tube flow results from this solution to the Navier–Stokes equations. Numerous experiments performed to substantiate the theoretical results show that the theory and experiment are in agreement for the laminar flow of Newtonian fluids in circular tubes or pipes. The flow remains laminar for Reynolds numbers, $\text{Re} = \rho V(2R)/\mu$, below 2100. Turbulent flow in tubes is considered in Chapter 8.

2.1.4 Steady, Axial, Laminar Flow in an Annulus

The differential equations (Eqs. 2.31, 2.32, 2.33) used in the preceding section for flow in a tube also apply to the axial flow in the annular space between two fixed, concentric cylinders (Fig. 2.7). Equation 2.35 for the velocity distribution still applies, but for the stationary annulus the boundary conditions become $v_z = 0$ at $r = r_o$ and $v_z = 0$ for $r = r_i$. With these two conditions the constants c_1 and c_2 in Eq. 2.35 can be determined and the velocity distribution becomes

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[r^2 - r_o^2 + \frac{r_i^2 - r_o^2}{\ln(r_o/r_i)}\ln\frac{r}{r_o}\right] \qquad (2.43)$$

The corresponding volume rate of flow is

$$Q = \int_{r_i}^{r_o} v_z(2\pi r)\,dr = -\frac{\pi}{8\mu}\left(\frac{\partial p}{\partial z}\right)\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]$$

or in terms of the pressure drop, Δp, in length ℓ of the annulus

$$Q = \frac{\pi \Delta p}{8\mu\ell}\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right] \qquad (2.44)$$

The velocity at any radial location within the annular space can be obtained from Eq. 2.43 . The maximum velocity occurs at the radius $r = r_m$ where $\partial v_z/\partial r = 0$. Thus,

$$r_m = \left[\frac{r_o^2 - r_i^2}{2\ln(r_o/r_i)}\right]^{1/2} \qquad (2.45)$$

An inspection of this result shows that the maximum velocity does not occur at the midpoint of the annular space, but rather it occurs nearer the inner cylinder. The specific location depends on r_o and r_i.

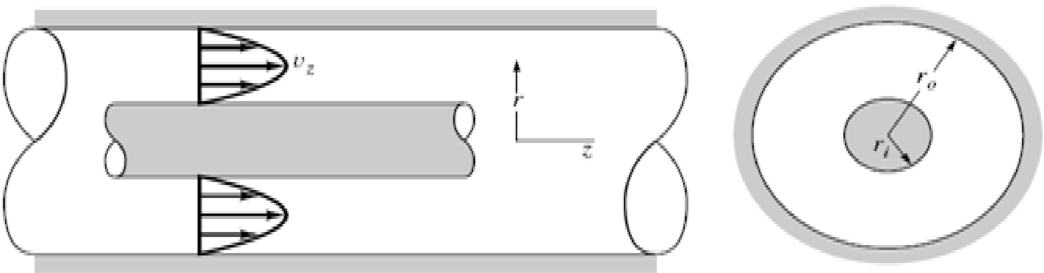
These results for flow through an annulus are only valid if the flow is laminar. A criterion based on the conventional Reynolds number (which is defined in terms of the tube diameter) cannot be directly applied to the annulus, since there are really "two" diameters involved. For tube cross sections other than simple circular tubes it is common practice to use an "effective" diameter, termed the *hydraulic diameter*, D_h, which is defined as

$$D_h = \frac{4 \times \text{cross-sectional area}}{\text{wetted perimeter}}$$

The wetted perimeter is the perimeter in contact with the fluid. For an annulus

$$D_h = \frac{4\pi(r_o^2 - r_i^2)}{2\pi(r_o + r_i)} = 2(r_o - r_i)$$

In terms of the hydraulic diameter, the Reynolds number is $Re = \rho D_h V/\mu$ (where $V = Q$/cross-sectional area), and it is commonly assumed that if this Reynolds number remains



■ FIGURE 2.7 **The viscous flow through an annulus.**

below 2100 the flow will be laminar. A further discussion of the concept of the hydraulic diameter as it applies to other noncircular cross sections is given in next sections

2.1.5 Flow between Long Concentric Cylinders:

Consider a fluid of constant (ρ, μ) between two concentric cylinders, as in Fig. 2.8 . There is no axial motion or end effect $v_z = \partial/\partial z = 0$. Let the inner cylinder rotate at angular velocity Ω_i. Let the outer cylinder be fixed. There is circular symmetry, so the velocity does not vary with θ and varies only with r.

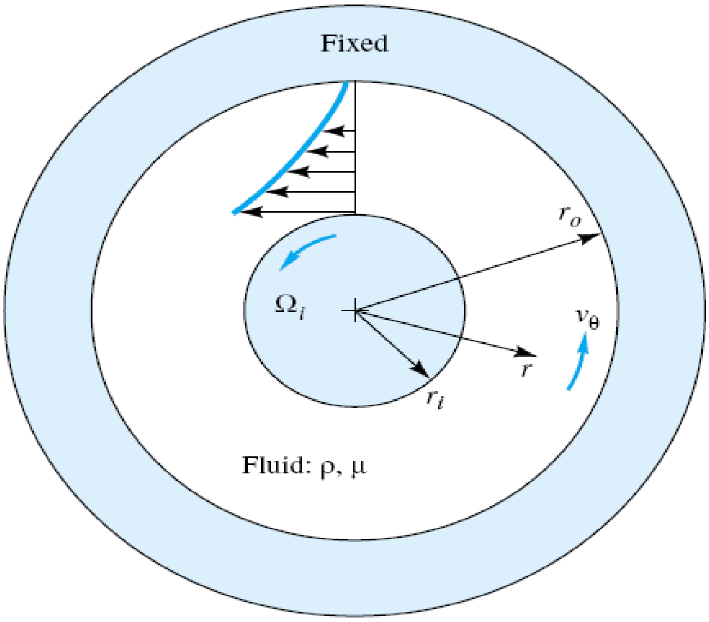


Fig. 2.8 Coordinate system for incompressible viscous flow between a fixed outer cylinder and a steadily rotating inner cylinder.

The continuity equation for this problem is Eq. (D.2):

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 = \frac{1}{r}\frac{d}{dr}(r v_r) \qquad \text{or} \qquad r v_r = \text{const}$$

Note that v_θ does not vary with θ. Since $v_r = 0$ at both the inner and outer cylinders, it follows that $v_r = 0$ everywhere and the motion can only be purely circumferential, $v_\theta = v_\theta(r)$. The θ-momentum equation (D.6) becomes

$$\rho(\mathbf{V}\cdot\nabla)v_\theta + \frac{\rho v_r v_\theta}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta + \mu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2}\right)$$

For the conditions of the present problem, all terms are zero except the last. Therefore the basic differential equation for flow between rotating cylinders is

$$\nabla^2 v_\theta = \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_\theta}{dr}\right) = \frac{v_\theta}{r^2} \qquad (2.46)$$

This is a linear second-order ordinary differential equation with the solution

$$v_\theta = C_1 r + \frac{C_2}{r}$$

The constants are found by the no-slip condition at the inner and outer cylinders:

Outer, at $r = r_o$: $$v_\theta = 0 = C_1 r_o + \frac{C_2}{r_o}$$

Inner, at $r = r_i$: $$v_\theta = \Omega_i r_i = C_1 r_i + \frac{C_2}{r_i}$$

The final solution for the velocity distribution is

Rotating inner cylinder: $$v_\theta = \Omega_i r_i \frac{r_o/r - r/r_o}{r_o/r_i - r_i/r_o} \qquad (2.47)$$

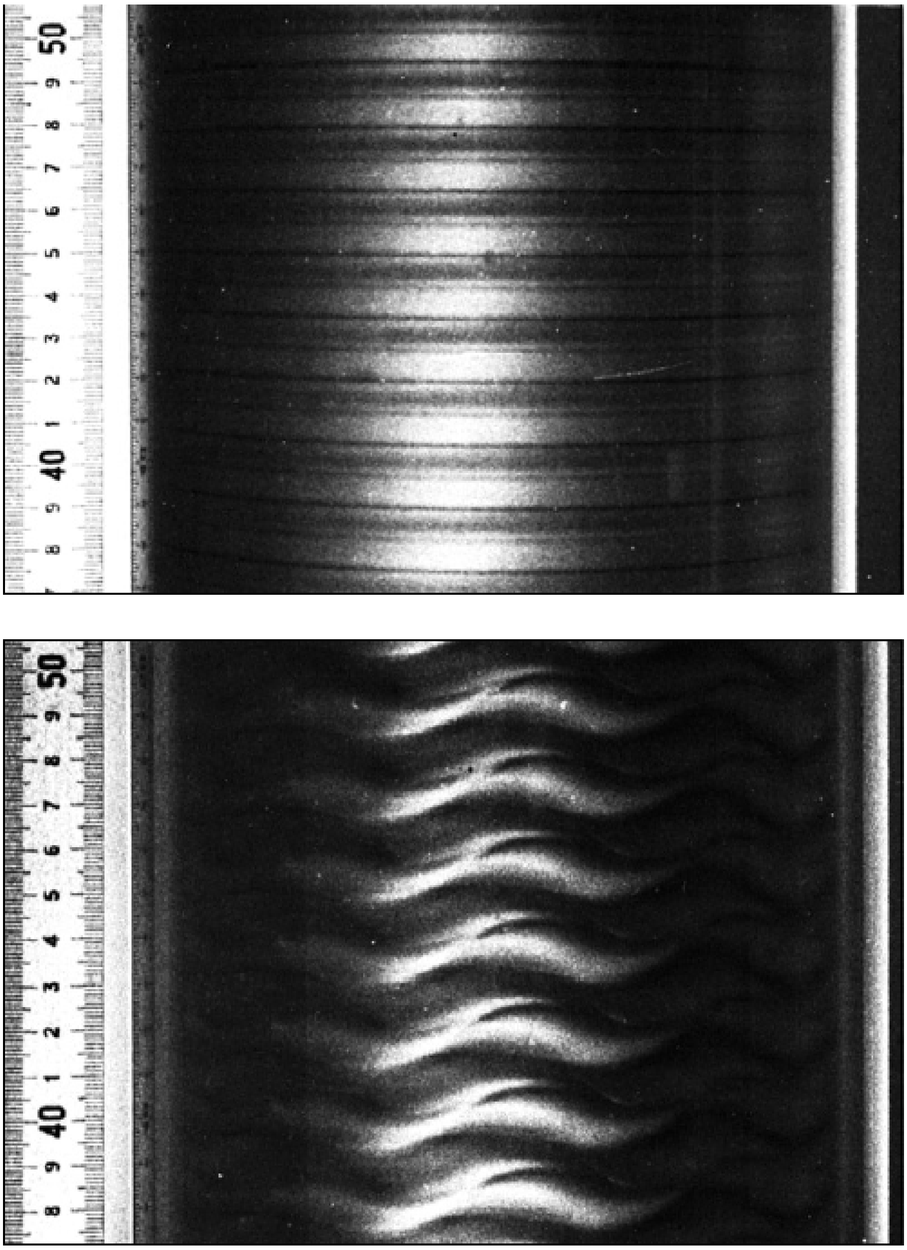
The velocity profile closely resembles the sketch in Fig. 2.8 . Variations of this case, such as a rotating outer cylinder, are given in the problem assignments.

The classic *Couette-flow* solution[11] of Eq. (2.47) describes a physically satisfying concave, two-dimensional, laminar-flow velocity profile as in Fig. 2.8 . The solution is mathematically exact for an incompressible fluid. However, it becomes unstable at a relatively low rate of rotation of the inner cylinder, as shown in 1923 in a classic paper by G. I. Taylor [17]. At a critical value of what is now called the *Taylor number*, denoted Ta,

$$\mathrm{Ta}_{\mathrm{crit}} = \frac{r_i(r_o - r_i)^3\Omega_i^2}{\nu^2} \approx 1700 \qquad (2.48)$$

the plane flow of Fig. 2.8 vanishes and is replaced by a laminar *three-dimensional* flow pattern consisting of rows of nearly square alternating toroidal vortices. An ex-

[11]Named after M. Couette, whose pioneering paper in 1890 established rotating cylinders as a method, still used today, for measuring the viscosity of fluids.



(*a*)

(*b*)

fig.2.9 Experimental verification of the instability of flow between a fixed outer and a rotating inner cylinder. (*a*) Toroidal Taylor vortices exist at 1.16 times the critical speed; (*b*) at 8.5 times the critical speed, the vortices are doubly periodic. (*After Koschmieder, Ref. 18.*) This instability does not occur if only the outer cylinder rotates.

perimental demonstration of toroidal "Taylor vortices" is shown in Fig. 2.9 *a*, measured at Ta ≈ 1.16 $\mathrm{Ta}_{\mathrm{crit}}$ by Koschmieder [18]. At higher Taylor numbers, the vortices also develop a circumferential periodicity but are still laminar, as illustrated in Fig. 2.9 *b*. At still higher Ta, turbulence ensues. This interesting instability reminds us that the Navier-Stokes equations, being nonlinear, do admit to multiple (nonunique) laminar solutions in addition to the usual instabilities associated with turbulence and chaotic dynamic systems.

Example 2.3:

A viscous liquid ($\rho = 1.18 \times 10^3$ kg/m^3; $\mu = 0.0045$ N · s/m^2) flows at a rate of 12 ml/s through a horizontal, 4-mm-diameter tube. (a) Determine the pressure drop along a l-m length of the tube which is far from the tube entrance so that the only component of velocity is parallel to the tube axis. (b) If a 2-mm-diameter rod is placed in the 4-mm-diameter tube to form a symmetric annulus, what is the pressure drop along a l-m length if the flowrate remains the same as in part (a)?

SOLUTION

(a) We first calculate the Reynolds number, Re, to determine whether or not the flow is laminar. The mean velocity is

$$V = \frac{Q}{(\pi/4)D^2} = \frac{(12 \text{ ml/s})(10^{-6} \text{ m}^3/\text{ml})}{(\pi/4)(4 \text{ mm} \times 10^{-3} \text{ m/mm})^2}$$
$$= 0.955 \text{ m/s}$$

and, therefore,

$$\text{Re} = \frac{\rho V D}{\mu} = \frac{(1.18 \times 10^3 \text{ kg/m}^3)(0.955 \text{ m/s})(4 \text{ mm} \times 10^{-3} \text{ m/mm})}{0.0045 \text{ N} \cdot \text{s/m}^2}$$
$$= 1000$$

Since the Reynolds number is well below the critical value of 2100 we can safely assume that the flow is laminar. Thus, we can apply Eq. 2.39 which gives for the pressure drop

$$\Delta p = \frac{8\mu \ell Q}{\pi R^4}$$
$$= \frac{8(0.0045 \text{ N} \cdot \text{s/m}^2)(1 \text{ m})(12 \times 10^{-6} \text{ m}^3/\text{s})}{\pi(2 \text{ mm} \times 10^{-3} \text{ m/mm})^4}$$
$$= 8.59 \text{ kPa} \qquad \text{(Ans)}$$

(b) For flow in the annulus, the mean velocity is

$$V = \frac{Q}{\pi(r_o^2 - r_i^2)} = \frac{12 \times 10^{-6} \text{ m}^3/\text{s}}{(\pi)[(2 \text{ mm} \times 10^{-3} \text{ m/mm})^2 - (1 \text{ mm} \times 10^{-3} \text{ m/mm})^2]}$$
$$= 1.27 \text{ m/s}$$

and the Reynolds number (based on the hydraulic diameter) is

$$\text{Re} = \frac{\rho 2(r_o - r_i)V}{\mu}$$
$$= \frac{(1.18 \times 10^3 \text{ kg/m}^3)(2)(2 \text{ mm} - 1 \text{ mm})(10^{-3} \text{ m/mm})(1.27 \text{ m/s})}{0.0045 \text{ N} \cdot \text{s/m}^2}$$
$$= 666$$

This value is also well below 2100 so the flow in the annulus should also be laminar. From Eq. 2.44 ,

$$\Delta p = \frac{8\mu\ell Q}{\pi}\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]^{-1}$$

so that

$$\Delta p = \frac{8(0.0045\ \text{N}\cdot\text{s/m}^2)(1\ \text{m})(12 \times 10^{-6}\ \text{m}^3/\text{s})}{\pi} \times \left\{(2 \times 10^{-3}\ \text{m})^4 - (1 \times 10^{-3}\ \text{m})^4 - \frac{[(2 \times 10^{-3}\ \text{m})^2 - (1 \times 10^{-3}\ \text{m})^2]^2}{\ln(2\ \text{mm}/1\ \text{mm})}\right\}^{-1}$$

$$= 68.2\ \text{kPa} \qquad \textbf{(Ans)}$$

The pressure drop in the annulus is much larger than that of the tube. This is not a surprising result, since to maintain the same flow in the annulus as that in the open tube the average velocity must be larger and the pressure difference along the annulus must overcome the shearing stresses that develop along both an inner and an outer wall. Even an annulus with a very small inner diameter will have a pressure drop significantly higher than that of an open tube. For example, if the inner diameter is only 1/100 of the outer diameter, Δp (annulus)/Δp (tube) = 1.28.

Example 2.4:

The accepted transition Reynolds number for flow in a circular pipe is $\text{Re}_{d,\text{crit}} \approx 2300$. For flow through a 5-cm-diameter pipe, at what velocity will this occur at 20°C for (*a*) airflow and (*b*) water flow?

Solution

Almost all pipe-flow formulas are based on the *average* velocity $V = Q/A$, not centerline or any other point velocity. Thus transition is specified at $\rho Vd/\mu \approx 2300$. With d known, we introduce the appropriate fluid properties at 20°C from Tables A.3 and A.4:

(*a*) Air: $$\frac{\rho Vd}{\mu} = \frac{(1.205\ \text{kg/m}^3)V(0.05\ \text{m})}{1.80\ \text{E-5 kg/(m}\cdot\text{s)}} = 2300 \quad \text{or} \quad V \approx 0.7\ \frac{\text{m}}{\text{s}}$$

(*b*) Water: $$\frac{\rho Vd}{\mu} = \frac{(998\ \text{kg/m}^3)V(0.05\ \text{m})}{0.001\ \text{kg/(m}\cdot\text{s)}} = 2300 \quad \text{or} \quad V = 0.046\ \frac{\text{m}}{\text{s}}$$

These are very low velocities, so most engineering air and water pipe flows are turbulent, not laminar. We might expect laminar duct flow with more viscous fluids such as lubricating oils or glycerin.

2.2 Flow Through An Inclined Circular Pipe:

a) Method of Using the Moody Chart:

As our next example of a specific viscous-flow analysis, we take the classic problem of flow in a full pipe, driven by pressure or gravity or both. Figure 2.10 shows the geometry of the pipe of radius R. The x-axis is taken in the flow direction and is inclined to the horizontal at an angle ϕ.

Before proceeding with a solution to the equations of motion, we can learn a lot by making a control-volume analysis of the flow between sections 1 and 2 in Fig. 2.10 . The continuity relation, reduces to

$$Q_1 = Q_2 = \text{const}$$

or

$$V_1 = \frac{Q_1}{A_1} = V_2 = \frac{Q_2}{A_2}$$

since the pipe is of constant area. The steady-flow energy equation reduces to

$$\frac{p_1}{\rho} + \frac{1}{2}\alpha_1 V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}\alpha_2 V_2^2 + gz_2 + gh_f \qquad (2.49)$$

since there are no shaft-work or heat-transfer effects. Now assume that the flow is fully

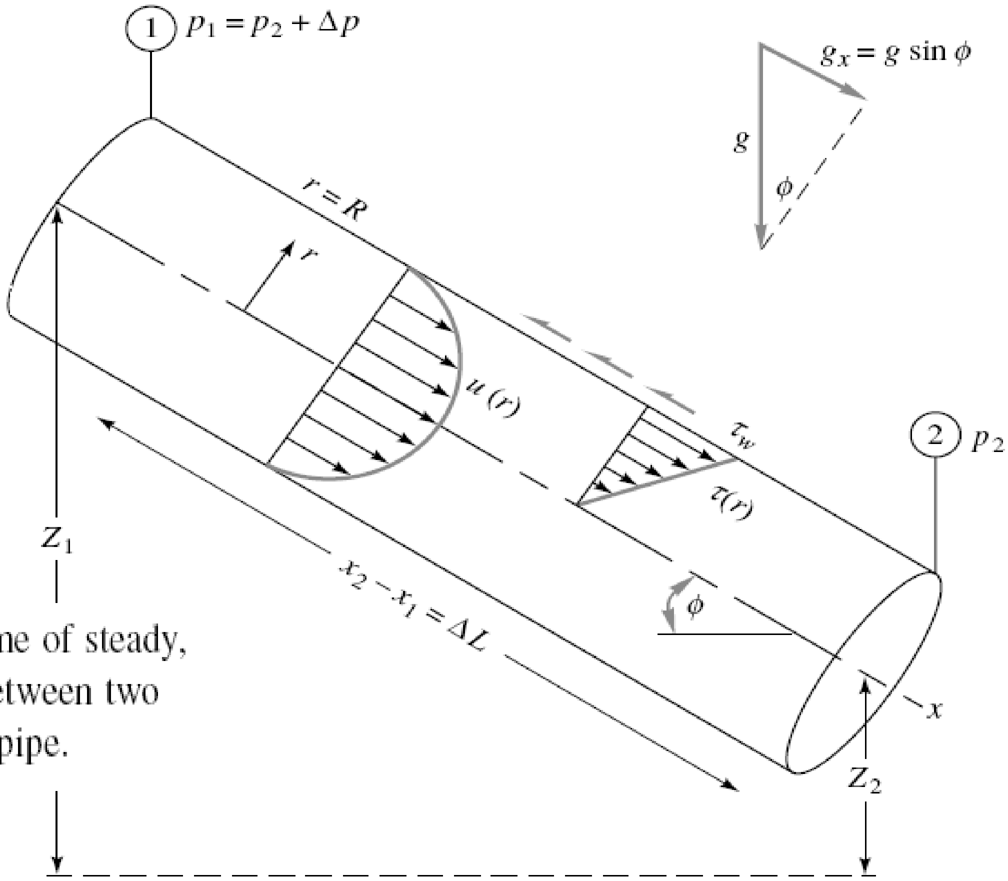


Fig. 2.10 Control volume of steady, fully developed flow between two sections in an inclined pipe.

developed (Fig. 2.6), and correct later for entrance effects. Then the kinetic-energy correction factor $\alpha_1 = \alpha_2$, and since $V_1 = V_2$, Eq. 2.49 now reduces to a simple expression for the friction-head loss h_f

$$h_f = \left(z_1 + \frac{p_1}{\rho g}\right) - \left(z_2 + \frac{p_2}{\rho g}\right) = \Delta\left(z + \frac{p}{\rho g}\right) = \Delta z + \frac{\Delta p}{\rho g} \qquad (2.50)$$

The pipe-head loss equals the change in the sum of pressure and gravity head, i.e., the change in height of the hydraulic grade line (HGL). Since the velocity head is constant through the pipe, h_f also equals the height change of the energy grade line (EGL). Recall that the EGL decreases downstream in a flow with losses unless it passes through an energy source, e.g., as a pump or heat exchanger.

Finally apply the momentum relation to the control volume in Fig. 2.10, accounting for applied forces due to pressure, gravity, and shear

$$\Delta p \; \pi R^2 + \rho g(\pi R^2)\,\Delta L \sin\phi - \tau_w(2\pi R)\,\Delta L = \dot{m}(V_2 - V_1) = 0 \qquad (2.51)$$

This equation relates h_f to the wall shear stress

$$\Delta z + \frac{\Delta p}{\rho g} = h_f = \frac{2\tau_w}{\rho g}\frac{\Delta L}{R} \qquad (2.52)$$

where we have substituted $\Delta z = \Delta L \sin\phi$ from Fig. 2.10.

So far we have not assumed either laminar or turbulent flow. If we can correlate τ_w with flow conditions, we have solved the problem of head loss in pipe flow. Functionally, we can assume that

$$\tau_w = F(\rho, V, \mu, d, \epsilon) \qquad (2.53)$$

where ϵ is the wall-roughness height. Then dimensional analysis tells us that

$$\frac{8\tau_w}{\rho V^2} = f = F\left(\text{Re}_d, \frac{\epsilon}{d}\right) \tag{2.54}$$

The dimensionless parameter f is called the *Darcy friction factor,* after Henry Darcy (1803–1858), a French engineer whose pipe-flow experiments in 1857 first established the effect of roughness on pipe resistance.

Combining Eqs. (2.52) and (2.54), we obtain the desired expression for finding pipe-head loss

$$h_f = f\,\frac{L}{d}\,\frac{V^2}{2g} \tag{2.55}$$

This is the *Darcy-Weisbach equation,* valid for duct flows of any cross section and for laminar and turbulent flow. It was proposed by Julius Weisbach, a German professor who in 1850 published the first modern textbook on hydrodynamics.

Our only remaining problem is to find the form of the function F in Eq. (2.54) and plot it in the Moody chart

b) Method of Solving the Equations of Motions:

For either laminar or turbulent flow, the continuity equation in cylindrical coordinates is given by

$$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\upsilon_\theta) + \frac{\partial u}{\partial x} = 0 \tag{2.56}$$

We assume that there is no swirl or circumferential variation, $\upsilon_\theta = \partial/\partial\theta = 0$, and fully developed flow: $u = u(r)$ only. Then Eq. (2.56) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) = 0$$

or

$$r\upsilon_r = \text{const} \tag{2.57}$$

But at the wall, $r = R$, $\upsilon_r = 0$ (no slip); therefore (2.57) implies that $\upsilon_r = 0$ everywhere. Thus in fully developed flow there is only one velocity component, $u = u(r)$.

The momentum differential equation in cylindrical coordinates now reduces to

$$\rho u\,\frac{\partial u}{\partial x} = -\frac{dp}{dx} + \rho g_x + \frac{1}{r}\frac{\partial}{\partial r}(r\tau) \tag{2.58}$$

where τ can represent either laminar or turbulent shear. But the left-hand side vanishes because $u = u(r)$ only. Rearrange, noting from Fig. 2.10 that $g_x = g \sin\phi$:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tau) = \frac{d}{dx}(p - \rho g x \sin\phi) = \frac{d}{dx}(p + \rho g z) \tag{2.59}$$

Since the left-hand side varies only with r and the right-hand side varies only with x, it follows that both sides must be equal to the same constant.[2] Therefore we can integrate Eq. (2.59) to find the shear distribution across the pipe, utilizing the fact that $\tau = 0$ at $r = 0$

$$\tau = \frac{1}{2}r\,\frac{d}{dx}(p + \rho g z) = (\text{const})(r) \tag{2.60}$$

[2]Ask your instructor to explain this to you if necessary.

Thus the shear varies linearly from the centerline to the wall, for either laminar or turbulent flow. This is also shown in Fig. 2.10. At $r = R$, we have the wall shear

$$\tau_w = \frac{1}{2}R\,\frac{\Delta p + \rho g\,\Delta z}{\Delta L} \tag{2.61}$$

which is identical with our momentum relation (2.52). We can now complete our study of pipe flow by applying either laminar or turbulent assumptions to fill out Eq. (2.60).

Laminar Flow Solution:

Note in Eq. (2.60) that the HGL slope $d(p + \rho gz)/dx$ is *negative* because both pressure and height drop with x. For laminar flow, $\tau = \mu\, du/dr$, which we substitute in Eq. (2.60)

$$\mu \frac{du}{dr} = \frac{1}{2} rK \qquad K = \frac{d}{dx}(p + \rho gz) \tag{2.62}$$

Integrate once

$$u = \frac{1}{4} r^2 \frac{K}{\mu} + C_1 \tag{2.63}$$

The constant C_1 is evaluated from the no-slip condition at the wall: $u = 0$ at $r = R$

$$0 = \frac{1}{4} R^2 \frac{K}{\mu} + C_1 \tag{2.64}$$

or $C_1 = -\frac{1}{4}R^2K/\mu$. Introduce into Eq. (2.63) to obtain the exact solution for laminar fully developed pipe flow

$$u = \frac{1}{4\mu}\left[-\frac{d}{dx}(p + \rho gz)\right](R^2 - r^2) \tag{2.65}$$

The laminar-flow profile is thus a paraboloid falling to zero at the wall and reaching a maximum at the axis

$$u_{\max} = \frac{R^2}{4\mu}\left[-\frac{d}{dx}(p + \rho gz)\right] \tag{2.66}$$

It resembles the sketch of $u(r)$ given in Fig. 2.10.

The laminar distribution (2.65) is called *Hagen-Poiseuille flow* to commemorate the experimental work of G. Hagen in 1839 and J. L. Poiseuille in 1940, both of whom established the pressure-drop law. The first theoretical derivation of Eq. (2.65) was given independently by E. Hagenbach and by F. Neumann around 1859.

Other pipe-flow results follow immediately from Eq. (2.65). The volume flow is

$$Q = \int_0^R u\, dA = \int_0^R u_{\max}\left(1 - \frac{r^2}{R^2}\right) 2\pi r\, dr$$

$$= \frac{1}{2} u_{\max} \pi R^2 = \frac{\pi R^4}{8\mu}\left[-\frac{d}{dx}(p + \rho gz)\right] \tag{2.67}$$

Thus the average velocity in laminar flow is one-half the maximum velocity

$$V = \frac{Q}{A} = \frac{Q}{\pi R^2} = \frac{1}{2} u_{\max} \tag{2.68}$$

For a horizontal tube ($\Delta z = 0$), Eq. (2.67) is of the form predicted by Hagen's experiment,

$$\Delta p = \frac{8\mu LQ}{\pi R^4} \tag{2.69}$$

The wall shear is computed from the wall velocity gradient

$$\tau_w = \left|\mu \frac{du}{dr}\right|_{r=R} = \frac{2\mu u_{\max}}{R} = \frac{1}{2} R \left|\frac{d}{dx}(p + \rho gz)\right| \tag{2.70}$$

This gives an exact theory for laminar Darcy friction factor

$$f = \frac{8\tau_w}{\rho V^2} = \frac{8(8\mu V/d)}{\rho V^2} = \frac{64\mu}{\rho Vd}$$

or

$$f_{\text{lam}} = \frac{64}{\text{Re}_d} \tag{2.71}$$

This is plotted later in the Moody chart (Fig. 2.13). The fact that f drops off with increasing Re_d should not mislead us into thinking that shear decreases with velocity: Eq. (2.70) clearly shows that τ_w is proportional to $u_{\max}$; it is interesting to note that τ_w is independent of density because the fluid acceleration is zero.

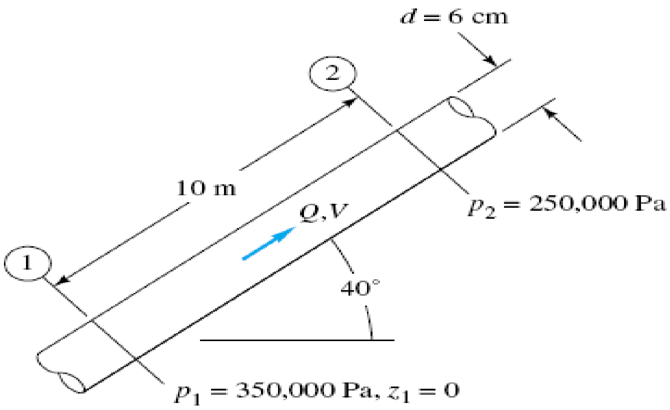
The laminar head loss follows from Eq. (2.55)

$$h_{f,\text{lam}} = \frac{64\mu}{\rho Vd} \frac{L}{d} \frac{V^2}{2g} = \frac{32\mu LV}{\rho g d^2} = \frac{128\mu LQ}{\pi \rho g d^4} \tag{2.72}$$

We see that laminar head loss is proportional to V.

Example 2.5:

An oil with $\rho = 900$ kg/m^3 and $\nu = 0.0002$ m^2/s flows upward through an inclined pipe as shown in Fig.E 2.5. The pressure and elevation are known at sections 1 and 2, 10 m apart. Assuming



E 2.5

steady laminar flow, (*a*) verify that the flow is up, (*b*) compute h_f between 1 and 2, and compute (*c*) Q, (*d*) V, and (*e*) Re_d. Is the flow really laminar?

Solution

Part (a) For later use, calculate

$$\mu = \rho\nu = (900 \text{ kg/m}^3)(0.0002 \text{ m}^2/\text{s}) = 0.18 \text{ kg/(m}\cdot\text{s)}$$

$$z_2 = \Delta L \sin 40° = (10 \text{ m})(0.643) = 6.43 \text{ m}$$

The flow goes in the direction of falling HGL; therefore compute the hydraulic grade-line height at each section

$$\text{HGL}_1 = z_1 + \frac{p_1}{\rho g} = 0 + \frac{350{,}000}{900(9.807)} = 39.65 \text{ m}$$

$$\text{HGL}_2 = z_2 + \frac{p_2}{\rho g} = 6.43 + \frac{250{,}000}{900(9.807)} = 34.75 \text{ m}$$

The HGL is lower at section 2; hence the flow is from 1 to 2 as assumed. *Ans. (a)*

Part (b) The head loss is the change in HGL:

$$h_f = \text{HGL}_1 - \text{HGL}_2 = 39.65 \text{ m} - 34.75 \text{ m} = 4.9 \text{ m}$$ *Ans. (b)*

Half the length of the pipe is quite a large head loss.

Part (c) We can compute Q from the various laminar-flow formulas, notably Eq. (2.72)

$$Q = \frac{\pi\rho g d^4 h_f}{128\mu L} = \frac{\pi(900)(9.807)(0.06)^4(4.9)}{128(0.18)(10)} = 0.0076 \text{ m}^3/\text{s}$$ *Ans. (c)*

Part (d) Divide Q by the pipe area to get the average velocity

$$V = \frac{Q}{\pi R^2} = \frac{0.0076}{\pi(0.03)^2} = 2.7 \text{ m/s}$$ *Ans. (d)*

Part (e) With V known, the Reynolds number is

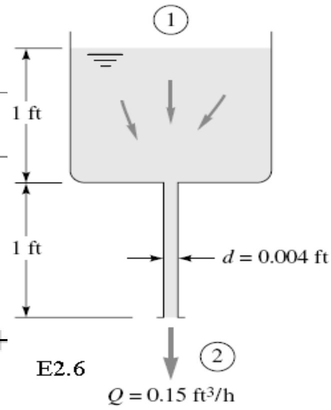
$$\text{Re}_d = \frac{Vd}{\nu} = \frac{2.7(0.06)}{0.0002} = 810$$ *Ans. (e)*

This is well below the transition value $\text{Re}_d = 2300$, and so we are fairly certain the flow is laminar.

Notice that by sticking entirely to consistent SI units (meters, seconds, kilograms, newtons) for all variables we avoid the need for any conversion factors in the calculations.

Example 2.6:

A liquid of specific weight $\rho g = 58$ lb/ft^3 flows by gravity through a 1-ft tank and a 1-ft capillary tube at a rate of 0.15 ft^3/h, as shown in Fig. E2.6. Sections 1 and 2 are at atmospheric pressure. Neglecting entrance effects, compute the viscosity of the liquid.



E2.6

Solution

Apply the steady-flow energy equation (2.49), including the correction factor α:

$$\frac{p_1}{\rho g} + \frac{\alpha_1 V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{\alpha_2 V_2^2}{2g} + z_2 +$$

The average exit velocity V_2 can be found from the volume flow and the pipe size:

$$V_2 = \frac{Q}{A_2} = \frac{Q}{\pi R^2} = \frac{(0.15/3600)\text{ ft}^3/\text{s}}{\pi(0.002\text{ ft})^2} \approx 3.32 \text{ ft/s}$$

Meanwhile $p_1 = p_2 = p_a$, and $V_1 \approx 0$ in the large tank. Therefore, approximately,

$$h_f \approx z_1 - z_2 - \alpha_2 \frac{V_2^2}{2g} = 2.0 \text{ ft} - 2.0 \frac{(3.32 \text{ ft/s})^2}{2(32.2 \text{ ft/s}^2)} \approx 1.66 \text{ ft}$$

where we have introduced $\alpha_2 = 2.0$ for laminar pipe flow. Note that h_f includes the entire 2-ft drop through the system and not just the 1-ft pipe length.

With the head loss known, the viscosity follows from our laminar-flow formula (2.72):

$$h_f = 1.66 \text{ ft} = \frac{32\mu L V}{\rho g d^2} = \frac{32\mu(1.0\text{ ft})(3.32\text{ ft/s})}{(58\text{ lbf/ft}^3)(0.004\text{ ft})^2} = 114{,}500\,\mu$$

or

$$\mu = \frac{1.66}{114{,}500} = 1.45 \text{ E-5 slug/(ft} \cdot \text{s)} \qquad \textit{Ans.}$$

Note that L in this formula is the *pipe* length of 1 ft. Finally, check the Reynolds number:

$$\text{Re}_d = \frac{\rho V d}{\mu} = \frac{(58/32.2\text{ slug/ft}^3)(3.32\text{ ft/s})(0.004\text{ ft})}{1.45\text{ E-5 slug/(ft}\cdot\text{s)}} = 1650 \qquad \text{laminar}$$

Since this is less than 2300, we conclude that the flow is indeed laminar. Actually, for this head loss, there is a *second* (turbulent) solution, as we shall see in Example 2.9.

2.2 *Case of Turbulent Flow* (this part may be omitted without loss of continuity)

2.2.1 *Semi-empirical Turbulent Shear Correlations:*

Throughout this chapter we assume constant density and viscosity and no thermal interaction, so that only the continuity and momentum equations are to be solved for velocity and pressure

Continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum: $$\rho \frac{d\mathbf{V}}{dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V} \qquad (2.73)$$

subject to no slip at the walls and known inlet and exit conditions. (We shall save our free-surface solutions for Chap. 10.)

Both laminar and turbulent flows satisfy Eqs. (2.73). For laminar flow, where there are no random fluctuations, we go right to the attack and solve them for a variety of geometries (as we did before in Sec. 2.1) leaving many more, of course, for the problems.

2.2.2 Reynolds Time-Averaging Concept:

For turbulent flow, because of the fluctuations, every velocity and pressure term in Eqs. 2.73 is a rapidly varying random function of time and space. At present our mathematics cannot handle such instantaneous fluctuating variables. No single pair of random functions $\mathbf{V}(x, y, z, t)$ and $p(x, y, z, t)$ is known to be a solution to Eqs. 2.73 . Moreover, our attention as engineers is toward the average or *mean* values of velocity, pressure, shear stress, etc., in a high-Reynolds-number (turbulent) flow. This approach led Osborne Reynolds in 1895 to rewrite Eqs. (2.73) in terms of mean or time-averaged turbulent variables.

The time mean $\overline{u}$ of a turbulent function $u(x, y, z, t)$ is defined by

$$\overline{u} = \frac{1}{T}\int_0^T u\, dt \tag{2.74}$$

where T is an averaging period taken to be longer than any significant period of the fluctuations themselves. The mean values of turbulent velocity and pressure are illustrated in Fig.2.11. For turbulent gas and water flows, an averaging period $T \approx 5$ s is usually quite adequate.

The *fluctuation* u' is defined as the deviation of u from its average value

$$u' = u - \overline{u} \tag{2.75}$$

also shown in Fig.2.11. It follows by definition that a fluctuation has zero mean value

$$\overline{u'} = \frac{1}{T}\int_0^T (u - \overline{u})\, dt = \overline{u} - \overline{u} = 0 \tag{2.76}$$

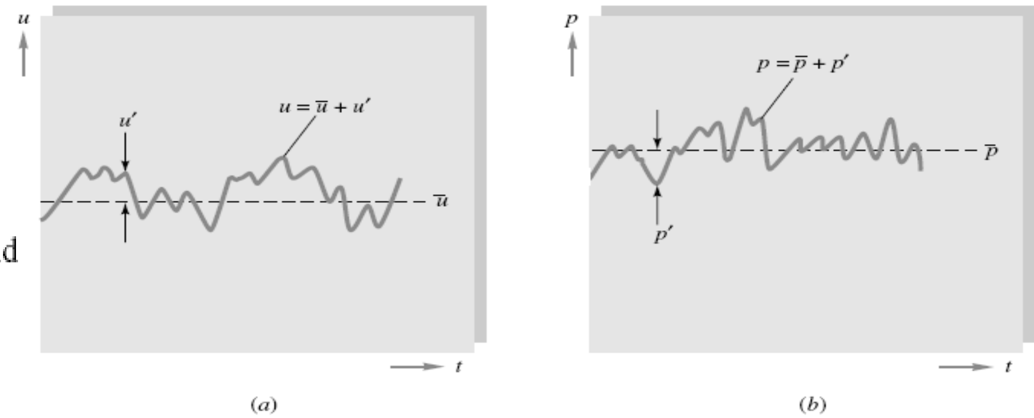


Fig.2.11 Definition of mean and fluctuating turbulent variables: (*a*) velocity; (*b*) pressure.

However, the mean square of a fluctuation is not zero and is a measure of the *intensity* of the turbulence

$$\overline{u'^2} = \frac{1}{T}\int_0^T u'^2\, dt \neq 0 \tag{2.77}$$

Nor in general are the mean fluctuation products such as $\overline{u'v'}$ and $\overline{u'p'}$ zero in a typical turbulent flow.

Reynolds' idea was to split each property into mean plus fluctuating variables

$$u = \overline{u} + u' \quad v = \overline{v} + v' \quad w = \overline{w} + w' \quad p = \overline{p} + p' \tag{2.78}$$

Substitute these into Eqs. 2.73 , and take the time mean of each equation. The continuity relation reduces to

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \qquad (2.79)$$

which is no different from a laminar continuity relation.

However, each component of the momentum equation 2.73b , after time averaging, will contain mean values plus three mean products, or *correlations,* of fluctuating velocities. The most important of these is the momentum relation in the mainstream, or x direction, which takes the form

$$\rho \frac{d\bar{u}}{dt} = -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial}{\partial x}\left(\mu \frac{\partial \bar{u}}{\partial x} - \rho \overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial \bar{u}}{\partial z} - \rho \overline{u'w'}\right) \qquad (2.80)$$

The three correlation terms $-\rho\overline{u'^2}$, $-\rho\overline{u'v'}$, and $-\rho\overline{u'w'}$ are called *turbulent stresses* because they have the same dimensions and occur right alongside the newtonian (laminar) stress terms $\mu(\partial \bar{u}/\partial x)$, etc. Actually, they are convective acceleration terms (which is why the density appears), not stresses, but they have the mathematical effect of stress and are so termed almost universally in the literature.

The turbulent stresses are unknown a priori and must be related by experiment to

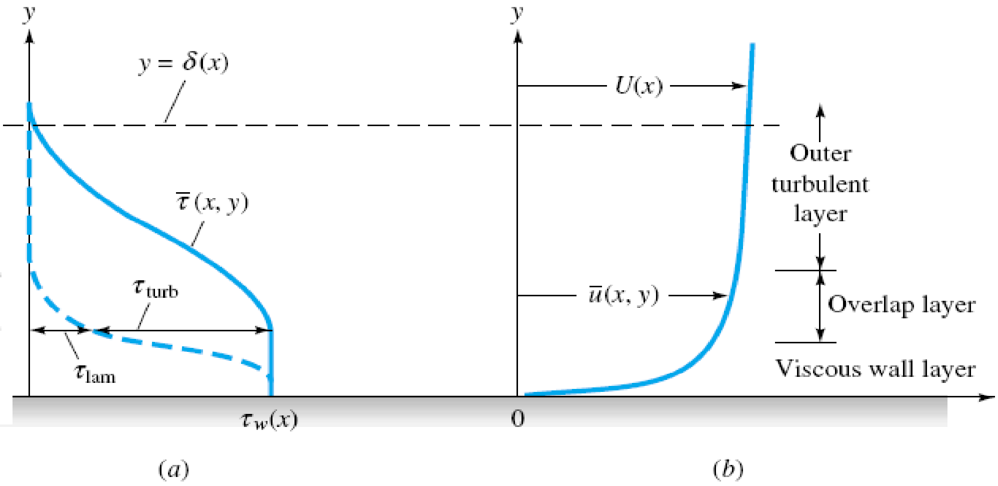


Fig.2.12 Typical velocity and shear distributions in turbulent flow near a wall: (*a*) shear; (*b*) velocity.

geometry and flow conditions, as detailed in Refs. 1 to 3. Fortunately, in duct and boundary-layer flow, the stress $-\rho\overline{u'v'}$ associated with direction y normal to the wall is dominant, and we can approximate with excellent accuracy a simpler streamwise momentum equation

$$\rho \frac{d\bar{u}}{dt} \approx -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial \tau}{\partial y} \qquad (2.81)$$

where

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} = \tau_{lam} + \tau_{turb} \qquad (2.82)$$

Figure 2.12 shows the distribution of τ_{lam} and τ_{turb} from typical measurements across a turbulent-shear layer near a wall. Laminar shear is dominant near the wall (the *wall layer*), and turbulent shear dominates in the *outer layer*. There is an intermediate region, called the *overlap layer,* where both laminar and turbulent shear are important. These three regions are labeled in Fig. 2.12.

In the outer layer τ_{turb} is two or three orders of magnitude greater than τ_{lam}, and vice versa in the wall layer. These experimental facts enable us to use a crude but very effective model for the velocity distribution $\bar{u}(y)$ across a turbulent wall layer.

The Logarithmic-Overlap Law:

We have seen in Fig. 2.12 that there are three regions in turbulent flow near a wall:

1. Wall layer: Viscous shear dominates.
2. Outer layer: Turbulent shear dominates.
3. Overlap layer: Both types of shear are important.

From now on let us agree to drop the overbar from velocity $\overline{u}$. Let τ_w be the wall shear stress, and let δ and U represent the thickness and velocity at the edge of the outer layer, $y = \delta$.

For the wall layer, Prandtl deduced in 1930 that u must be independent of the shear-layer thickness

$$u = f(\mu, \tau_w, \rho, y) \tag{2.83}$$

By dimensional analysis, this is equivalent to

$$u^+ = \frac{u}{u^*} = F\left(\frac{yu^*}{\nu}\right) \qquad u^* = \left(\frac{\tau_w}{\rho}\right)^{1/2} \tag{2.84}$$

Equation (2.84) is called the *law of the wall,* and the quantity u^* is termed the *friction velocity* because it has dimensions $\{LT^{-1}\}$, although it is not actually a flow velocity.

Subsequently, Kármán in 1933 deduced that u in the outer layer is independent of molecular viscosity, but its deviation from the stream velocity U must depend on the layer thickness δ and the other properties

$$(U - u)_{\text{outer}} = g(\delta, \tau_w, \rho, y) \tag{2.85}$$

Again, by dimensional analysis we rewrite this as

$$\frac{U - u}{u^*} = G\left(\frac{y}{\delta}\right) \tag{2.86}$$

where u^* has the same meaning as in Eq. (2.84). Equation (2.86) is called the *velocity-defect law* for the outer layer.

Both the wall law (2.84) and the defect law (2.86) are found to be accurate for a wide variety of experimental turbulent duct and boundary-layer flows [1 to 3]. They are different in form, yet they must overlap smoothly in the intermediate layer. In 1937 C. B. Millikan showed that this can be true only if the overlap-layer velocity varies logarithmically with y:

$$\frac{u}{u^*} = \frac{1}{\kappa} \ln \frac{yu^*}{\nu} + B \qquad \text{overlap layer} \tag{2.87}$$

Over the full range of turbulent smooth wall flows, the dimensionless constants κ and B are found to have the approximate values $\kappa \approx 0.41$ and $B \approx 5.0$. Equation (2.87) is called the *logarithmic-overlap layer.*

Thus by dimensional reasoning and physical insight we infer that a plot of u versus $\ln y$ in a turbulent-shear layer will show a curved wall region, a curved outer region, and a straight-line logarithmic overlap. Figure 2.13 shows that this is exactly the case. The four outer-law profiles shown all merge smoothly with the logarithmic-overlap law but have different magnitudes because they vary in external pressure gradient. The wall law is unique and follows the linear viscous relation

$$u^+ = \frac{u}{u^*} = \frac{yu^*}{\nu} = y^+ \tag{2.88}$$

from the wall to about $y^+ = 5$, thereafter curving over to merge with the logarithmic law at about $y^+ = 30$.

Believe it or not, Fig.2.13, which is nothing more than a shrewd correlation of velocity profiles, is the basis for most existing "theory" of turbulent-shear flows. Notice that we have not solved any equations at all but have merely expressed the streamwise velocity in a neat form.

There is serendipity in Fig.2.13: The logarithmic law (2.87), instead of just being a short overlapping link, actually approximates nearly the entire velocity profile, except for the outer law when the pressure is increasing strongly downstream (as in a diffuser). The inner-wall law typically extends over less than 2 percent of the profile and can be neglected. Thus we can use Eq. (2.87) as an excellent approximation to solve

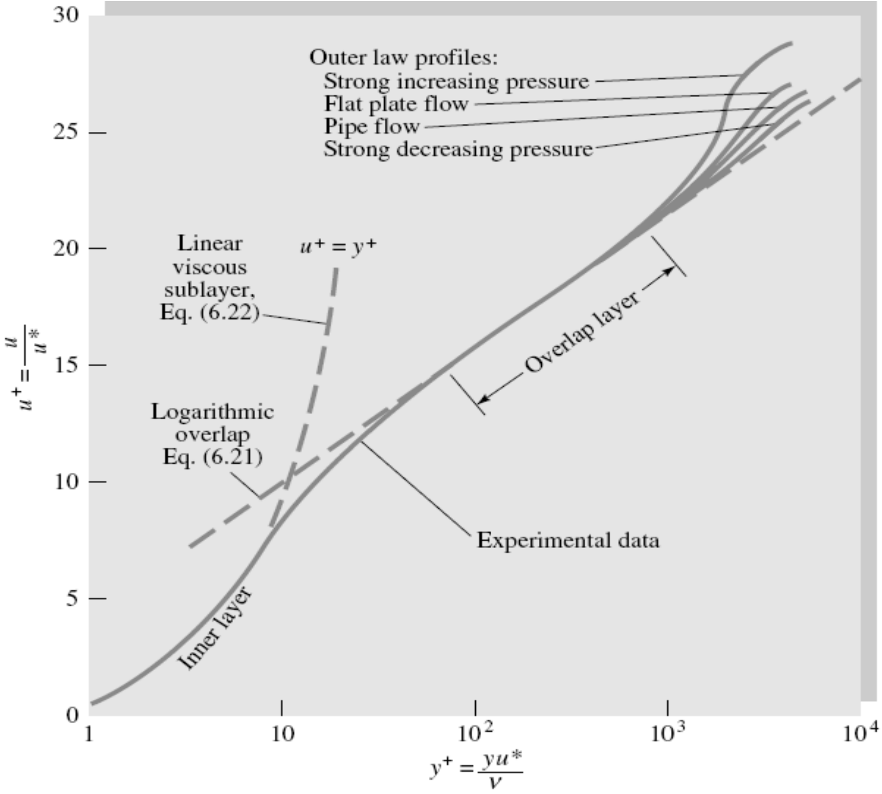
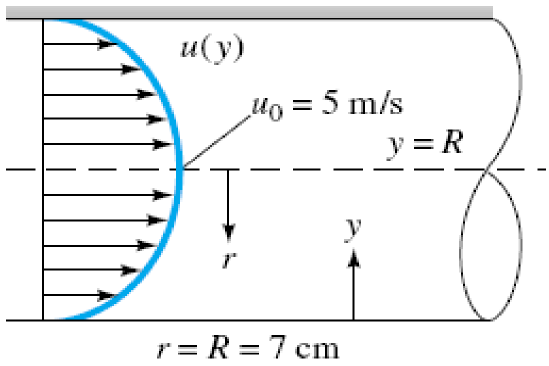


Fig2.13 Experimental verification of the inner-, outer-, and overlap-layer laws relating velocity profiles in turbulent wall flow.



E 2.7

nearly every turbulent-flow problem presented in this and the next chapter. Many additional applications are given in Refs. 2 and 3.

Example 2.7:

Air at 20°C flows through a 14-cm-diameter tube under fully developed conditions. The centerline velocity is $u_0 = 5$ m/s. Estimate from Fig.2.13 (*a*) the friction velocity u^*, (*b*) the wall shear str Part a) (*c*) the average velocity $V = Q/A$.

Solution

For pipe flow Fig.2.13 shows that the logarithmic law, Eq. (2.87), is accurate all the way to the center of the tube. From Fig. E2.7 $y = R - r$ should go from the wall to the centerline as shown. At the center $u = u_0$, $y = R$, and Eq. (6.21) becomes

$$\frac{u_0}{u^*} = \frac{1}{0.41} \ln \frac{Ru^*}{\nu} + 5.0 \qquad (1)$$

Since we know that $u_0 = 5$ m/s and $R = 0.07$ m, u^* is the only unknown in Eq. (1). Find the solution by trial and error

$$u^* = 0.228 \text{ m/s} = 22.8 \text{ cm/s} \qquad \textit{Ans. (a)}$$

where we have taken $\nu = 1.51 \times 10^{-5}$ m²/s for air from Table 1.4.

Part (b) Assuming a pressure of 1 atm, we have $\rho = p/(RT) = 1.205$ kg/m^3. Since by definition $u^* = (\tau_w/\rho)^{1/2}$, we compute

$$\tau_w = \rho u^{*2} = (1.205 \text{ kg/m}^3)(0.228 \text{ m/s})^2 = 0.062 \text{ kg/(m} \cdot \text{s}^2) = 0.062 \text{ Pa} \qquad \textit{Ans. (b)}$$

This is a very small shear stress, but it will cause a large pressure drop in a long pipe (170 Pa for every 100 m of pipe).

Part (c) The average velocity V is found by integrating the logarithmic-law velocity distribution

$$V = \frac{Q}{A} = \frac{1}{\pi R^2}\int_0^R u\, 2\pi r\, dr \tag{2}$$

Introducing $u = u^*[(1/\kappa)\ln(yu^*/\nu) + B]$ from Eq. (2.87) and noting that $y = R - r$, we can carry out the integration of Eq. (2), which is rather laborious. The final result is

$$V = 0.835u_0 = 4.17 \text{ m/s} \qquad \textit{Ans. (c)}$$

We shall not bother showing the integration here because it is all worked out and a very neat formula is given in Eqs. 2.115 and 2.125.

Notice that we started from almost nothing (the pipe diameter and the centerline velocity) and found the answers without solving the differential equations of continuity and momentum. We just used the logarithmic law, Eq. (2.87), which makes the differential equations unnecessary for pipe flow. This is a powerful technique, but you should remember that all we are doing is using an experimental velocity correlation to approximate the actual solution to the problem.

We should check the Reynolds number to ensure turbulent flow

$$\text{Re}_d = \frac{Vd}{\nu} = \frac{(4.17 \text{ m/s})(0.14 \text{ m})}{1.51 \times 10^{-5} \text{ m}^2/\text{s}} = 38{,}700$$

Since this is greater than 4000, the flow is definitely turbulent.

Turbulent Flow Solutions:

For turbulent pipe flow we need not solve a differential equation but instead proceed with the logarithmic law, as in Example 2.7. Assume that Eq. (2.87) correlates the local mean velocity $u(r)$ all the way across the pipe

$$\frac{u(r)}{u^*} \approx \frac{1}{\kappa}\ln\frac{(R-r)u^*}{\nu} + B \tag{2.89}$$

where we have replaced y by $R - r$. Compute the average velocity from this profile

$$V = \frac{Q}{A} = \frac{1}{\pi R^2}\int_0^R u^*\left[\frac{1}{\kappa}\ln\frac{(R-r)u^*}{\nu} + B\right]2\pi r\, dr$$

$$= \frac{1}{2}u^*\left(\frac{2}{\kappa}\ln\frac{Ru^*}{\nu} + 2B - \frac{3}{\kappa}\right) \tag{2.90}$$

Introducing $\kappa = 0.41$ and $B = 5.0$, we obtain, numerically,

$$\frac{V}{u^*} \approx 2.44\ln\frac{Ru^*}{\nu} + 1.34 \tag{2.91}$$

This looks only marginally interesting until we realize that V/u^* is directly related to the Darcy friction factor

$$\frac{V}{u^*} = \left(\frac{\rho V^2}{\tau_w}\right)^{1/2} = \left(\frac{8}{f}\right)^{1/2} \qquad (2.92)$$

Moreover, the argument of the logarithm in (2.91) is equivalent to

$$\frac{Ru^*}{\nu} = \frac{\frac{1}{2}Vd}{\nu}\frac{u^*}{V} = \frac{1}{2}\mathrm{Re}_d\left(\frac{f}{8}\right)^{1/2} \qquad (2.93)$$

Introducing (2.93) and (2.92) into Eq. (2.91), changing to a base-10 logarithm, and rearranging, we obtain

$$\frac{1}{f^{1/2}} \approx 1.99 \log\,(\mathrm{Re}_d f^{1/2}) - 1.02 \qquad (2.94)$$

In other words, by simply computing the mean velocity from the logarithmic-law correlation, we obtain a relation between the friction factor and Reynolds number for turbulent pipe flow. Prandtl derived Eq. (2.94) in 1935 and then adjusted the constants slightly to fit friction data better

$$\frac{1}{f^{1/2}} = 2.0 \log\,(\mathrm{Re}_d f^{1/2}) - 0.8 \qquad (2.95)$$

This is the accepted formula for a smooth-walled pipe. Some numerical values may be listed as follows:

Re_d	4000	10^4	10^5	10^6	10^7	10^8
f	0.0399	0.0309	0.0180	0.0116	0.0081	0.0059

Thus f drops by only a factor of 5 over a 10,000-fold increase in Reynolds number. Equation (2.95) is cumbersome to solve if Re_d is known and f is wanted. There are many alternate approximations in the literature from which f can be computed explicitly from Re_d

$$f = \begin{cases} 0.316\,\mathrm{Re}_d^{-1/4} & 4000 < \mathrm{Re}_d < 10^5 \quad \text{H. Blasius (1911)} \\ \left(1.8 \log \dfrac{\mathrm{Re}_d}{6.9}\right)^{-2} & \text{Ref. 9} \end{cases} \qquad (2.96)$$

Blasius, a student of Prandtl, presented his formula in the first correlation ever made of pipe friction versus Reynolds number. Although his formula has a limited range, it illustrates what was happening to Hagen's 1839 pressure-drop data. For a horizontal pipe, from Eq. (2.96),

$$h_f = \frac{\Delta p}{\rho g} = f\frac{L}{d}\frac{V^2}{2g} \approx 0.316\left(\frac{\mu}{\rho V d}\right)^{1/4}\frac{L}{d}\frac{V^2}{2g}$$

or

$$\Delta p \approx 0.158\, L\rho^{3/4}\mu^{1/4}d^{-5/4}V^{7/4} \qquad (2.97)$$

at low turbulent Reynolds numbers. This explains why Hagen's data for pressure drop begin to increase as the 1.75 power of the velocity, in Fig. 2.8 . Note that Δp varies only slightly with viscosity, which is characteristic of turbulent flow. Introducing $Q = \frac{1}{4}\pi d^2 V$ into Eq. (2.97), we obtain the alternate form

$$\Delta p \approx 0.241 L\rho^{3/4}\mu^{1/4}d^{-4.75}Q^{1.75} \qquad (2.98)$$

For a given flow rate Q, the turbulent pressure drop decreases with diameter even more sharply than the laminar formula (2.72). Thus the quickest way to reduce required pumping pressure is to increase the pipe size, although, of course, the larger pipe is more expensive. Doubling the pipe size decreases Δp by a factor of about 27 for a given Q.

The maximum velocity in turbulent pipe flow is given by Eq. (2.89), evaluated at $r = 0$

$$\frac{u_{max}}{u^*} \approx \frac{1}{\kappa} \ln \frac{Ru^*}{\nu} + B \qquad (2.99)$$

Combining this with Eq. (2.90), we obtain the formula relating mean velocity to maximum velocity

$$\frac{V}{u_{max}} \approx (1 + 1.33\sqrt{f})^{-1} \qquad (2.100)$$

Some numerical values are

Re_d	4000	10^4	10^5	10^6	10^7	10^8
V/u_{max}	0.790	0.811	0.849	0.875	0.893	0.907

The ratio varies with the Reynolds number and is much larger than the value of 0.5 predicted for all laminar pipe flow in Eq. (2.68). Thus a turbulent velocity profile, as shown in Fig. 2.14 , is very flat in the center and drops off sharply to zero at the wall.

Effect of Rough Walls:

It was not known until experiments in 1800 by Coulomb [6] that surface roughness has an effect on friction resistance. It turns out that the effect is negligible for laminar pipe flow, and all the laminar formulas derived in this section are valid for rough walls also. But turbulent flow is strongly affected by roughness. In Fig. 2.13 the linear viscous sublayer only extends out to $y^+ = yu^*/\nu = 5$. Thus, compared with the diameter, the sublayer thickness y_s is only

$$\frac{y_s}{d} = \frac{5\nu/u^*}{d} = \frac{14.1}{Re_d f^{1/2}} \qquad (2.101)$$

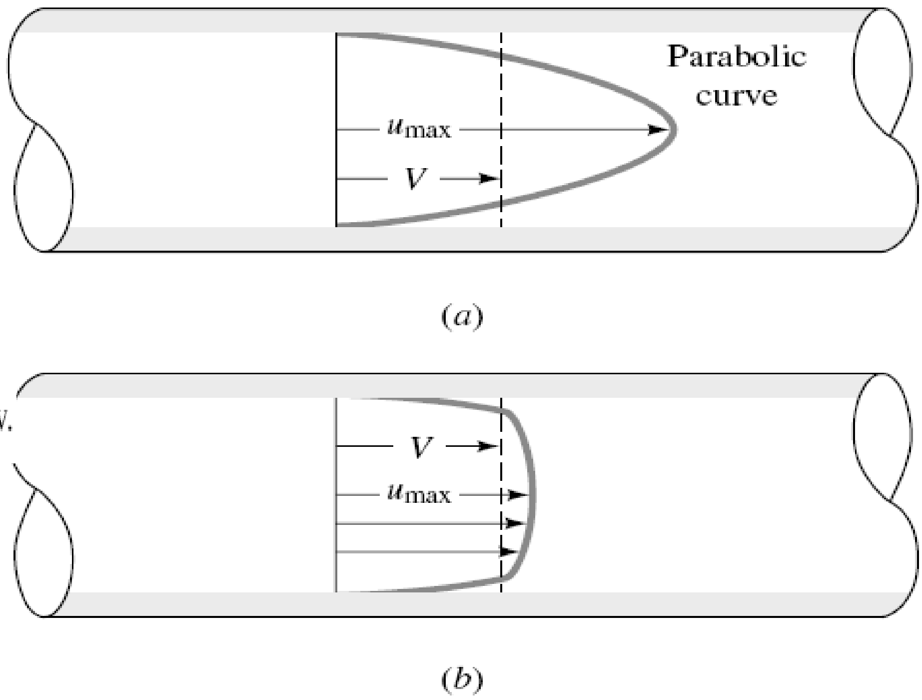
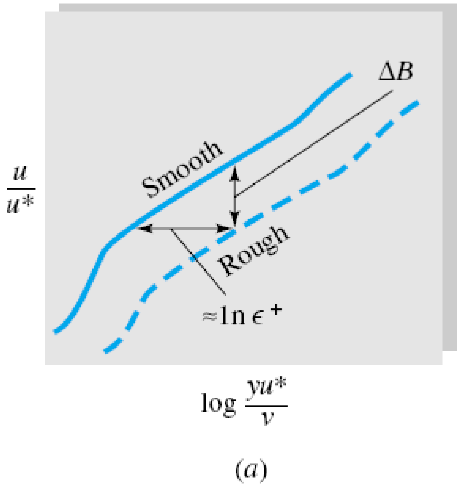


Fig. 2.14 Comparison of laminar and turbulent pipe-flow velocity profiles for the same volume flow: (a) laminar flow; (b) turbulent flow.



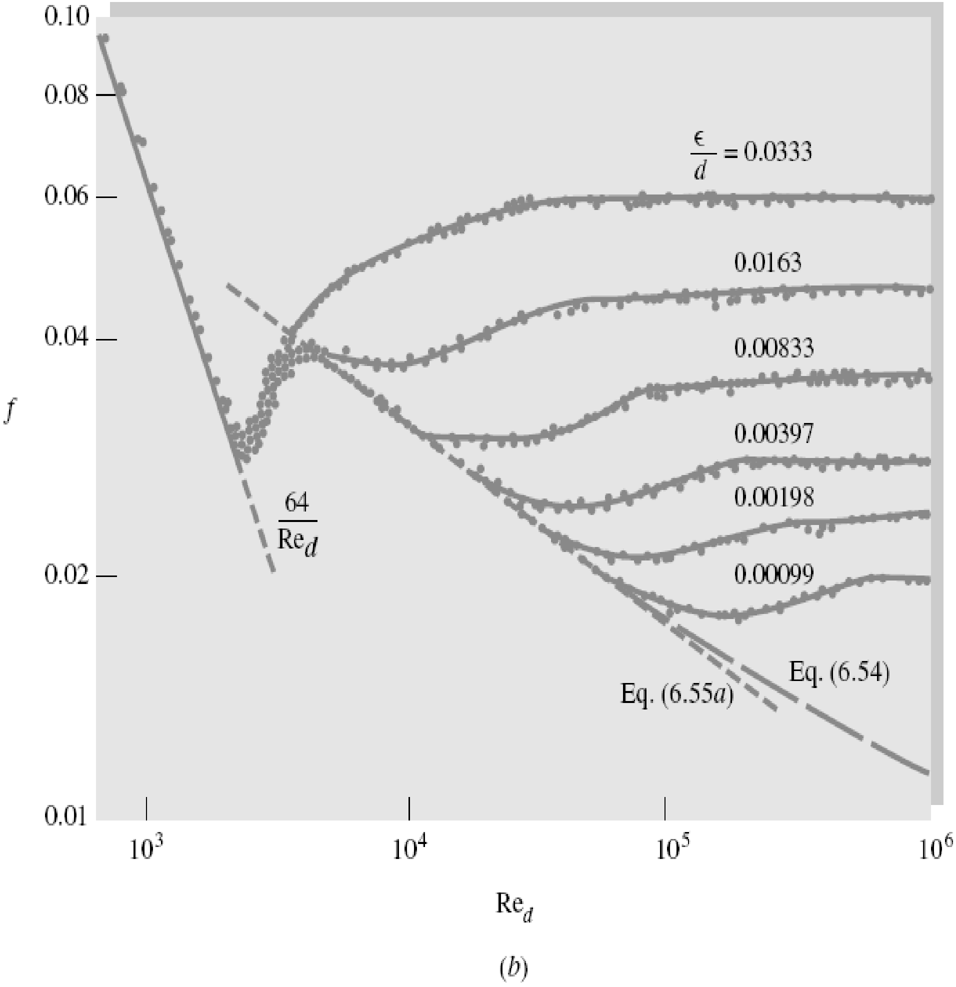


Fig. 2.15 Effect of wall roughness on turbulent pipe flow. (*a*) The logarithmic overlap-velocity profile shifts down and to the right; (*b*) experiments with sand-grain roughness by Nikuradse [7] show a systematic increase of the turbulent friction factor with the roughness ratio.

For example, at $\mathrm{Re}_d = 10^5$, $f = 0.0180$, and $y_s/d = 0.001$, a wall roughness of about $0.001d$ will break up the sublayer and profoundly change the wall law in Fig.2.13.

Measurements of $u(y)$ in turbulent rough-wall flow by Prandtl's student Nikuradse [7] show, as in Fig. 2.15*a*, that a roughness height ϵ will force the logarithm-law profile outward on the abscissa by an amount approximately equal to ln ϵ^+, where $\epsilon^+ = \epsilon u^*/\nu$. The slope of the logarithm law remains the same, $1/\kappa$, but the shift outward causes the constant B to be less by an amount $\Delta B \approx (1/\kappa) \ln \epsilon^+$.

Nikuradse [7] simulated roughness by gluing uniform sand grains onto the inner walls of the pipes. He then measured the pressure drops and flow rates and correlated friction factor versus Reynolds number in Fig. 2.15*b*. We see that laminar friction is unaffected, but turbulent friction, after an *onset* point, increases monotonically with the roughness ratio ϵ/d. For any given ϵ/d, the friction factor becomes constant *(fully rough)* at high Reynolds numbers. These points of change are certain values of $\epsilon^+ = \epsilon u^*/\nu$:

$\dfrac{\epsilon u^*}{\nu} < 5$: *hydraulically smooth* walls, no effect of roughness on friction

$5 \le \dfrac{\epsilon u^*}{\nu} \le 70$: *transitional* roughness, moderate Reynolds-number effect

$\dfrac{\epsilon u^*}{\nu} > 70$: *fully rough* flow, sublayer totally broken up and friction independent of Reynolds number

For fully rough flow, $\epsilon^+ > 70$, the log-law downshift ΔB in Fig. 2.15 *a* is

$$\Delta B \approx \frac{1}{\kappa} \ln \epsilon^+ - 3.5 \qquad 2.102$$

and the logarithm law modified for roughness becomes

$$u^+ = \frac{1}{\kappa}\ln y^+ + B - \Delta B = \frac{1}{\kappa}\ln\frac{y}{\epsilon} + 8.5 \tag{2.103}$$

The viscosity vanishes, and hence fully rough flow is independent of the Reynolds number. If we integrate Eq. (2.103) to obtain the average velocity in the pipe, we obtain

$$\frac{V}{u*} = 2.44 \ln\frac{d}{\epsilon} + 3.2$$

or

$$\frac{1}{f^{1/2}} = -2.0 \log\frac{\epsilon/d}{3.7} \quad \text{fully rough flow} \tag{2.104}$$

There is no Reynolds-number effect; hence the head loss varies exactly as the square of the velocity in this case. Some numerical values of friction factor may be listed:

ϵ/d	0.00001	0.0001	0.001	0.01	0.05
f	0.00806	0.0120	0.0196	0.0379	0.0716

The friction factor increases by 9 times as the roughness increases by a factor of 5000. In the transitional-roughness region, sand grains behave somewhat differently from commercially rough pipes, so Fig. 2.15b has now been replaced by the Moody chart.

The Moody Chart:

In 1939 to cover the transitionally rough range, Colebrook [9] combined the smooth-wall [Eq. (2.95)] and fully rough [Eq. (2.104)] relations into a clever interpolation formula

$$\frac{1}{f^{1/2}} = -2.0 \log\left(\frac{\epsilon/d}{3.7} + \frac{2.51}{\text{Re}_d f^{1/2}}\right) \tag{2.105}$$

This is the accepted design formula for turbulent friction. It was plotted in 1944 by Moody [8] into what is now called the *Moody chart* for pipe friction (Fig. 2.16). The Moody chart is probably the most famous and useful figure in fluid mechanics. It is accurate to ± 15 percent for design calculations over the full range shown in Fig. 2.16 . It can be used for circular and noncircular (Sec. 2.3) pipe flows and for open-channel flows (Chap. 10). The data can even be adapted as an approximation to boundary-layer flows (part 4).

Equation (2.105) is cumbersome to evaluate for f if Re_d is known, although it easily yields to the EES Equation Solver. An alternate explicit formula given by Haaland [33] as

$$\frac{1}{f^{1/2}} \approx -1.8 \log\left[\frac{6.9}{\text{Re}_d} + \left(\frac{\epsilon/d}{3.7}\right)^{1.11}\right] \tag{2.105a}$$

varies less than 2 percent from Eq. (2.105).

The shaded area in the Moody chart indicates the range where transition from laminar to turbulent flow occurs. There are no reliable friction factors in this range, $2000 < \text{Re}_d < 4000$. Notice that the roughness curves are nearly horizontal in the fully rough regime to the right of the dashed line.

From tests with commercial pipes, recommended values for average pipe roughness are listed in Table 2.1 .

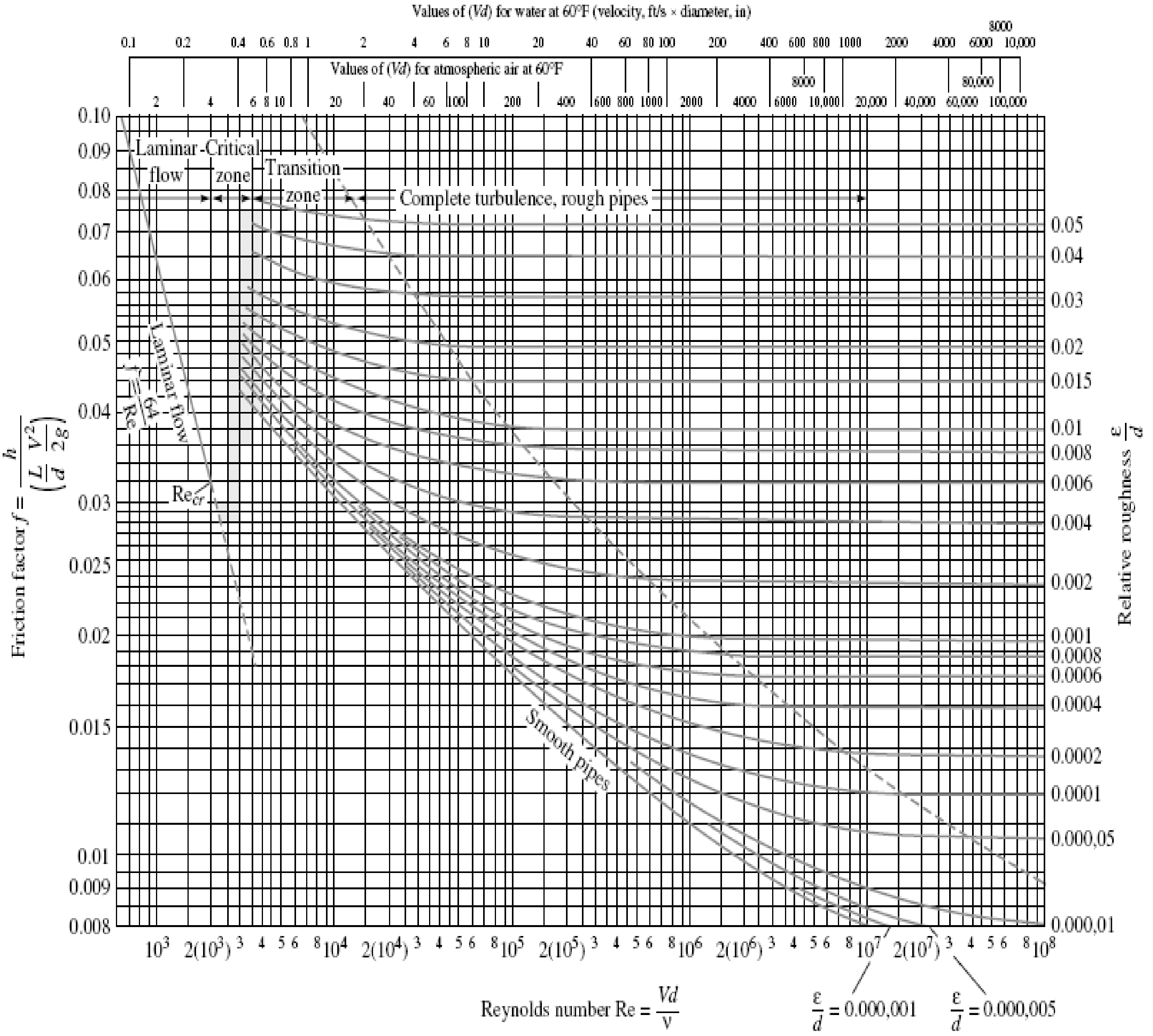


Fig. 2.16 The Moody chart for pipe friction with smooth and rough walls. This chart is identical to Eq. **2.105** for turbulent flow. *(From Ref. 8, by permission of the ASME.)*

Table 2.1 Recommended Roughness Values for Commercial Ducts

Material	Condition	ϵ ft	ϵ mm	Uncertainty, %
Steel	Sheet metal, new	0.00016	0.05	± 60
	Stainless, new	0.000007	0.002	± 50
	Commercial, new	0.00015	0.046	± 30
	Riveted	0.01	3.0	± 70
	Rusted	0.007	2.0	± 50
Iron	Cast, new	0.00085	0.26	± 50
	Wrought, new	0.00015	0.046	± 20
	Galvanized, new	0.0005	0.15	± 40
	Asphalted cast	0.0004	0.12	± 50
Brass	Drawn, new	0.000007	0.002	± 50
Plastic	Drawn tubing	0.000005	0.0015	± 60
Glass	—	Smooth	Smooth	
Concrete	Smoothed	0.00013	0.04	± 60
	Rough	0.007	2.0	± 50
Rubber	Smoothed	0.000033	0.01	± 60
Wood	Stave	0.0016	0.5	± 40

Example 2.8:

Compute the loss of head and pressure drop in 200 ft of horizontal 6-in-diameter asphalted cast-iron pipe carrying water with a mean velocity of 6 ft/s.

Solution

One can estimate the Reynolds number of water and air from the Moody chart. Look across the top of the chart to V (ft/s) $\times$ d (in) = 36, and then look directly down to the bottom abscissa to find that Re_d(water) $\approx 2.7 \times 10^5$. The roughness ratio for asphalted cast iron ($\epsilon = 0.0004$ ft) is

$$\frac{\epsilon}{d} = \frac{0.0004}{\frac{6}{12}} = 0.0008$$

Find the line on the right side for $\epsilon/d = 0.0008$, and follow it to the left until it intersects the vertical line for $Re = 2.7 \times 10^5$. Read, approximately, $f = 0.02$ [or compute $f = 0.0197$ from Eq. (2.105a)]. Then the head loss is

$$h_f = f\frac{L}{d}\frac{V^2}{2g} = (0.02)\frac{200}{0.5}\frac{(6\ \text{ft/s})^2}{2(32.2\ \text{ft/s}^2)} = 4.5\ \text{ft} \qquad \textit{Ans.}$$

The pressure drop for a horizontal pipe ($z_1 = z_2$) is

$$\Delta p = \rho g h_f = (62.4\ \text{lbf/ft}^3)(4.5\ \text{ft}) = 280\ \text{lbf/ft}^2 \qquad \textit{Ans.}$$

Moody points out that this computation, even for clean new pipe, can be considered accurate only to about ± 10 percent.

Example 2.9:

Oil, with $\rho = 900$ kg/m^3 and $\nu = 0.00001$ m^2/s, flows at 0.2 m^3/s through 500 m of 200-mm-diameter cast-iron pipe. Determine (*a*) the head loss and (*b*) the pressure drop if the pipe slopes down at 10° in the flow direction.

Solution

First compute the velocity from the known flow rate

$$V = \frac{Q}{\pi R^2} = \frac{0.2\ \text{m}^3/\text{s}}{\pi(0.1\ \text{m})^2} = 6.4\ \text{m/s}$$

Then the Reynolds number is

$$Re_d = \frac{Vd}{\nu} = \frac{(6.4\ \text{m/s})(0.2\ \text{m})}{0.00001\ \text{m}^2/\text{s}} = 128{,}000$$

From Table 2.1 , $\epsilon = 0.26$ mm for cast-iron pipe. Then

$$\frac{\epsilon}{d} = \frac{0.26\ \text{mm}}{200\ \text{mm}} = 0.0013$$

[3]This example was given by Moody in his 1944 paper [8].

Enter the Moody chart on the right at $\epsilon/d = 0.0013$ (you will have to interpolate), and move to the left to intersect with $Re = 128{,}000$. Read $f \approx 0.0225$ [from Eq. (2.105) for these values we could compute $f = 0.0227$]. Then the head loss is

$$h_f = f\frac{L}{d}\frac{V^2}{2g} = (0.0225)\frac{500\ \text{m}}{0.2\ \text{m}}\frac{(6.4\ \text{m/s})^2}{2(9.81\ \text{m/s}^2)} = 117\ \text{m} \qquad \textit{Ans. (a)}$$

From Eq. (2.51) for the inclined pipe,

$$h_f = \frac{\Delta p}{\rho g} + z_1 - z_2 = \frac{\Delta p}{\rho g} + L\sin 10°$$

or
$$\Delta p = \rho g[h_f - (500\ \text{m})\sin 10°] = \rho g(117\ \text{m} - 87\ \text{m})$$

$$= (900\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(30\ \text{m}) = 265{,}000\ \text{kg/(m}\cdot\text{s}^2) = 265{,}000\ \text{Pa} \qquad \textit{Ans. (b)}$$

Example 2.10:

Repeat Example 2.6 to see whether there is any possible turbulent-flow solution for a smooth-walled pipe.

Solution

In Example 2.6 we estimated a head loss $h_f \approx 1.66$ ft, assuming laminar exit flow ($\alpha \approx 2.0$). For this condition the friction factor is

$$f = h_f \frac{d}{L}\frac{2g}{V^2} = (1.66 \text{ ft})\frac{(0.004 \text{ ft})(2)(32.2 \text{ ft/s}^2)}{(1.0 \text{ ft})(3.32 \text{ ft/s})^2} \approx 0.0388$$

For laminar flow, $\text{Re}_d = 64/f = 64/0.0388 \approx 1650$, as we showed in Example 2.6 . However, from the Moody chart (Fig. 2.16), we see that $f = 0.0388$ also corresponds to a *turbulent* smooth-wall condition, at $\text{Re}_d \approx 4500$. If the flow actually were turbulent, we should change our kinetic-energy factor to $\alpha \approx 1.06$, whence the corrected $h_f \approx 1.82$ ft and $f \approx 0.0425$. With f known, we can estimate the Reynolds number from our formulas:

$$\text{Re}_d \approx 3250 \quad [\text{Eq. (2.95)}] \qquad \text{or} \qquad \text{Re}_d \approx 3400 \quad [\text{Eq. (2.96b)}]$$

So the flow *might* have been turbulent, in which case the viscosity of the fluid would have been

$$\mu = \frac{\rho V d}{\text{Re}_d} = \frac{1.80(3.32)(0.004)}{3250} = 7.2 \times 10^{-6} \text{ slug/(ft} \cdot \text{s)} \qquad \textit{Ans.}$$

This is about 55 percent less than our laminar estimate in Example 2.6 . The moral is to keep the capillary-flow Reynolds number below about 1000 to avoid such duplicate solutions.

Three Types of Pipe Flow Problems:

The Moody chart (Fig. 2.16) can be used to solve almost any problem involving friction losses in long pipe flows. However, many such problems involve considerable iteration and repeated calculations using the chart because the standard Moody chart is essentially a *head-loss chart.* One is supposed to know all other variables, compute Re_d, enter the chart, find f, and hence compute h_f. This is one of three fundamental problems which are commonly encountered in pipe-flow calculations:

1. Given d, L, and V or Q, ρ, μ, and g, compute the head loss h_f (head-loss problem).
2. Given d, L, h_f, ρ, μ, and g, compute the velocity V or flow rate Q (flow-rate problem).
3. Given Q, L, h_f, ρ, μ, and g, compute the diameter d of the pipe (sizing problem).

Only problem 1 is well suited to the Moody chart. We have to iterate to compute velocity or diameter because both d and V are contained in the ordinate *and* the abscissa of the chart.

There are two alternatives to iteration for problems of type 2 and 3: (a) preparation of a suitable new Moody-type chart (see Prob. 2.62 and 2.73); or (b) the use of *solver* software, especially the Engineering Equation Solver, known as EES [47], which gives the answer directly if the proper data are entered.

Even though velocity (or flow rate) appears in both the ordinate and the abscissa on the Moody chart, iteration for turbulent flow is nevertheless quite fast, because f varies so slowly with Re_d. Alternately, in the spirit of Example , we could change the scaling variables to (ρ, μ, d) and thus arrive at dimensionless head loss versus dimensionless *velocity*. The result is[4]

$$\zeta = \text{fcn}(Re_d) \qquad \text{where} \qquad \zeta = \frac{gd^3h_f}{L\nu^2} = \frac{f\,Re_d^2}{2} \tag{2.106}$$

Example did this and offered the simple correlation $\zeta \approx 0.155\,Re_d^{1.75}$, which is valid for turbulent flow with smooth walls and $Re_d \leq 1$ E5.

A formula valid for all turbulent pipe flows is found by simply rewriting the Colebrook interpolation, Eq. (2.105), in the form of Eq. (2.106):

$$Re_d = -(8\zeta)^{1/2}\log\left(\frac{\epsilon/d}{3.7} + \frac{1.775}{\sqrt{\zeta}}\right) \qquad \zeta = \frac{gd^3h_f}{L\nu^2} \tag{2.107}$$

Given ζ, we compute Re_d (and hence velocity) directly. Let us illustrate these two approaches with the following example.

Example 2.11:

Oil, with $\rho = 950$ kg/m^3 and $\nu = 2$ E-5 m^2/s, flows through a 30-cm-diameter pipe 100 m long with a head loss of 8 m. The roughness ratio is $\epsilon/d = 0.0002$. Find the average velocity and flow rate.

Direct Solution

First calculate the dimensionless head-loss parameter:

$$\zeta = \frac{gd^3h_f}{L\nu^2} = \frac{(9.81\ \text{m/s}^2)(0.3\ \text{m})^3(8.0\ \text{m})}{(100\ \text{m})(2\ \text{E-5}\ \text{m}^2/\text{s})^2} = 5.30\ \text{E7}$$

[4]The parameter ζ was suggested by H. Rouse in 1942.

Now enter Eq. 2.107 to find the Reynolds number:

$$Re_d = -[8(5.3\ \text{E7})]^{1/2}\log\left(\frac{0.0002}{3.7} + \frac{1.775}{\sqrt{5.3\ \text{E7}}}\right) = 72{,}600$$

The velocity and flow rate follow from the Reynolds number:

$$V = \frac{\nu\,Re_d}{d} = \frac{(2\ \text{E-5}\ \text{m}^2/\text{s})(72{,}600)}{0.3\ \text{m}} \approx 4.84\ \text{m/s}$$

$$Q = V\frac{\pi}{4}d^2 = \left(4.84\frac{\text{m}}{\text{s}}\right)\frac{\pi}{4}(0.3\ \text{m})^2 \approx 0.342\ \text{m}^3/\text{s} \qquad \textit{Ans.}$$

No iteration is required, but this idea falters if additional losses are present.

Iterative Solution

By definition, the friction factor is known except for V:

$$f = h_f\frac{d}{L}\frac{2g}{V^2} = (8\ \text{m})\left(\frac{0.3\ \text{m}}{100\ \text{m}}\right)\left[\frac{2(9.81\ \text{m/s}^2)}{V^2}\right] \qquad \text{or} \qquad fV^2 \approx 0.471 \qquad \text{(SI units)}$$

To get started, we only need to guess f, compute $V = \sqrt{0.471/f}$, then get Re_d, compute a better f from the Moody chart, and repeat. The process converges fairly rapidly. A good first guess is the "fully rough" value for $\epsilon/d = 0.0002$, or $f \approx 0.014$ from Fig. 2.16. The iteration would be as follows:

Guess $f \approx 0.014$, then $V = \sqrt{0.471/0.014} = 5.80$ m/s and $Re_d = Vd/\nu \approx 87{,}000$. At $Re_d = 87{,}000$ and $\epsilon/d = 0.0002$, compute $f_{new} \approx 0.0195$ [Eq. 2.105].

New $f \approx 0.0195$, $V = \sqrt{0.481/0.0195} = 4.91$ m/s and $Re_d = Vd/\nu = 73{,}700$. At $Re_d = 73{,}700$ and $\epsilon/d = 0.0002$, compute $f_{new} \approx 0.0201$ [Eq. 2.105].

Better $f \approx 0.0201$, $V = \sqrt{0.471/0.0201} = 4.84$ m/s and $Re_d \approx 72{,}600$. At $Re_d = 72{,}600$ and $\epsilon/d = 0.0002$, compute $f_{new} \approx 0.0201$ [Eq. 2.105].

We have converged to three significant figures. Thus our iterative solution is

$$V = 4.84 \text{ m/s}$$

$$Q = V\left(\frac{\pi}{4}\right)d^2 = (4.84)\left(\frac{\pi}{4}\right)(0.3)^2 \approx 0.342 \text{ m}^3/\text{s} \qquad \textit{Ans.}$$

The iterative approach is straightforward and not too onerous, so it is routinely used by engineers. Obviously this repetitive procedure is ideal for a personal computer.

Example 2.12:

Work Moody's problem (Example 2.8) backward, assuming that the head loss of 4.5 ft is known and the velocity (6.0 ft/s) is unknown.

Direct Solution

Find the parameter ζ, and compute the Reynolds number from Eq. 2.107 :

$$\zeta = \frac{gd^3h_f}{L\nu^2} = \frac{(32.2 \text{ ft/s}^2)(0.5 \text{ ft})^3(4.5 \text{ ft})}{(200 \text{ ft})(1.1 \text{ E-5 ft}^2/\text{s})^2} = 7.48 \text{ E8}$$

Eq. 2.107 : $$Re_d = -[8(7.48 \text{ E8})]^{1/2} \log\left(\frac{0.0008}{3.7} + \frac{1.775}{\sqrt{7.48 \text{ E8}}}\right) \approx 274{,}800$$

Then $$V = \nu\frac{Re_d}{d} = \frac{(1.1 \text{ E-5})(274{,}800)}{0.5} \approx 6.05 \text{ ft/s} \qquad \textit{Ans.}$$

We did not get 6.0 ft/s exactly because the 4.5-ft head loss was rounded off in Example 2.8 .

Iterative Solution

As in Eq. 2.75, the friction factor is related to velocity:

$$f = h_f\frac{d}{L}\frac{2g}{V^2} = (4.5 \text{ ft})\left(\frac{0.5 \text{ ft}}{200 \text{ ft}}\right)\left[\frac{2(32.2 \text{ ft/s}^2)}{V^2}\right] \approx \frac{0.7245}{V^2}$$

or $$V = \sqrt{0.7245/f}$$

Knowing $\epsilon/d = 0.0008$, we can guess f and iterate until the velocity converges. Begin with the fully rough estimate $f \approx 0.019$ from Fig. 2.16 . The resulting iterates are

$f_1 = 0.019$: $\quad V_1 = \sqrt{0.7245/f_1} = 6.18$ ft/s $\quad Re_{d_1} = \frac{Vd}{\nu} = 280{,}700$

$f_2 = 0.0198$: $\quad V_2 = 6.05$ ft/s $\quad Re_{d_2} = 274{,}900$

$f_3 = 0.01982$: $\quad V_3 = 6.046$ ft/s $\qquad$ *Ans.*

The calculation converges rather quickly to the same result as that obtained through direct computation.

Type 3- Problem: Find the Pipe Diameter:

The Moody chart is especially awkward for finding the pipe size, since d occurs in all three parameters f, Re_d, and ϵ/d. Further, it depends upon whether we know the velocity or the flow rate. We cannot know both, or else we could immediately compute $d = \sqrt{4Q/(\pi V)}$.

Let us assume that we know the flow rate Q. Note that this requires us to redefine the Reynolds number in terms of Q:

$$\mathrm{Re}_d = \frac{Vd}{\nu} = \frac{4Q}{\pi d \nu} \tag{2.108}$$

Then, if we choose (Q, ρ, μ) as scaling parameters (to eliminate d), we obtain the functional relationship

$$\mathrm{Re}_d = \frac{4Q}{\pi d \nu} = \mathrm{fcn}\left(\frac{g h_f}{L \nu^5}, \frac{\epsilon \nu}{Q}\right) \tag{2.109}$$

and can thus solve d when the right-hand side is known. Unfortunately, the writer knows of no *formula* for this relation, nor is he able to rearrange Eq. (2.105) into the explicit form of Eq. (2.109). One could recalculate and *plot* the relation, and indeed an ingenious "pipe-sizing" plot is given in Ref. 13. Here it seems reasonable to forgo a plot or curve fitted formula and to simply set up the problem as an iteration in terms of the Moody-chart variables. In this case we also have to set up the friction factor in terms

$$f = h_f \frac{d}{L}\frac{2g}{V^2} = \frac{\pi^2}{8}\frac{g h_f d^5}{L Q^2} \tag{2.110}$$

The following two examples illustrate the iteration.

Example 2.13:

Work Example 2.11backward, assuming that $Q = 0.342\ \mathrm{m^3/s}$ and $\epsilon = 0.06$ mm are known but that d (30 cm) is unknown. Recall $L = 100$ m, $\rho = 950\ \mathrm{kg/m^3}$, $\nu = 2\ \mathrm{E\text{-}5\ m^2/s}$, and $h_f = 8$ m.

Iterative Solution

First write the diameter in terms of the friction factor:

$$f = \frac{\pi^2}{8}\frac{(9.81\ \mathrm{m/s^2})(8\ \mathrm{m})d^5}{(100\ \mathrm{m})(0.342\ \mathrm{m^3/s})^2} = 8.28 d^5 \qquad \text{or} \qquad d \approx 0.655 f^{1/5} \tag{1}$$

in SI units. Also write the Reynolds number and roughness ratio in terms of the diameter:

$$\mathrm{Re}_d = \frac{4(0.342\ \mathrm{m^3/s})}{\pi(2\ \mathrm{E\text{-}5\ m^2/s})d} = \frac{21{,}800}{d} \tag{2}$$

$$\frac{\epsilon}{d} = \frac{6\ \mathrm{E\text{-}5\ m}}{d} \tag{3}$$

Guess f, compute d from (1), then compute Re_d from (2) and ϵ/d from (3), and compute a better f from the Moody chart or Eq. 2.105 . Repeat until (fairly rapid) convergence. Having no initial estimate for f, the writer guesses $f \approx 0.03$ (about in the middle of the turbulent portion of the Moody chart). The following calculations result:

$$f \approx 0.03 \qquad d \approx 0.655(0.03)^{1/5} \approx 0.325\ \mathrm{m}$$

$$\text{Re}_d \approx \frac{21{,}800}{0.325} \approx 67{,}000 \qquad \frac{\epsilon}{d} \approx 1.85\text{ E-4}$$

Eq. (2.95): $\quad f_{new} \approx 0.0203 \quad$ then $\quad d_{new} \approx 0.301$ m

$$\text{Re}_{d,new} \approx 72{,}500 \qquad \frac{\epsilon}{d} \approx 2.0\text{ E-4}$$

Eq. (2.95): $\quad f_{better} \approx 0.0201 \quad$ and $\quad d = 0.300$ m *Ans.*

The procedure has converged to the correct diameter of 30 cm given in Example2.11.

Example 2.14:

Work Moody's problem, Example 2.8, backward to find the unknown (6 in) diameter if the flow rate $Q = 1.18$ ft^3/s is known. Recall $L = 200$ ft, $\epsilon = 0.0004$ ft, and $\nu = 1.1$ E-5 ft^2/s.

Solution

Write f, Re_d, and ϵ/d in terms of the diameter:

$$f = \frac{\pi^2}{8}\frac{gh_f d^5}{LQ^2} = \frac{\pi^2}{8}\frac{(32.2\text{ ft/s}^2)(4.5\text{ ft})d^5}{(200\text{ ft})(1.18\text{ ft}^3/\text{s})^2} = 0.642d^5 \quad \text{or} \quad d \approx 1.093 f^{1/5} \qquad (1)$$

$$\text{Re}_d = \frac{4(1.18\text{ ft}^3/\text{s})}{\pi(1.1\text{ E-5 ft}^2/\text{s})\, d} = \frac{136{,}600}{d} \qquad (2)$$

$$\frac{\epsilon}{d} = \frac{0.0004\text{ ft}}{d} \qquad (3)$$

with everything in BG units, of course. Guess f; compute d from (1), Re_d from (2), and ϵ/d from (3); and then compute a better f from the Moody chart. Repeat until convergence. The writer traditionally guesses an initial $f \approx 0.03$:

$$f \approx 0.03 \qquad d \approx 1.093(0.03)^{1/5} \approx 0.542\text{ ft}$$

$$\text{Re}_d = \frac{136{,}600}{0.542} \approx 252{,}000 \qquad \frac{\epsilon}{d} \approx 7.38\text{ E-4}$$

$$f_{new} \approx 0.0196 \qquad d_{new} \approx 0.498\text{ ft} \qquad \text{Re}_d \approx 274{,}000 \qquad \frac{\epsilon}{d} \approx 8.03\text{ E-4}$$

$f_{better} \approx 0.0198 \qquad d \approx 0.499$ ft *Ans.*

Convergence is rapid, and the predicted diameter is correct, about 6 in. The slight discrepancy (0.499 rather than 0.500 ft) arises because h_f was rounded to 4.5 ft.

In discussing pipe-sizing problems, we should remark that commercial pipes are made only in certain sizes. Table 2.2 lists standard water-pipe sizes in the United States. If the sizing calculation gives an intermediate diameter, the next largest pipe size should be selected.

Table 2.2 Nominal and Actual Sizes of Schedule 40 Wrought-Steel Pipe*

Nominal size, in	Actual ID, in
$\frac{1}{8}$	0.269
$\frac{1}{4}$	0.364
$\frac{3}{8}$	0.493
$\frac{1}{2}$	0.622
$\frac{3}{4}$	0.824
1	1.049
$1\frac{1}{2}$	1.610
2	2.067
$2\frac{1}{2}$	2.469
3	3.068

*Nominal size within 1 percent for 4 in or larger.

Flow in Noncircular Ducts (the hydraulic diameter):

If the duct is noncircular, the analysis of fully developed flow follows that of the circular pipe but is more complicated algebraically. For laminar flow, one can solve the exact equations of continuity and momentum. For turbulent flow, the logarithm-law velocity profile can be used, or (better and simpler) the hydraulic diameter is an excellent approximation.

For a noncircular duct, the control-volume concept of Fig. 2.10 is still valid, but the cross-sectional area A does not equal πR^2 and the cross-sectional perimeter wetted by the shear stress $\mathscr{P}$ does not equal $2\pi R$. The momentum equation thus becomes

$$\Delta p\, A + \rho g A\, \Delta L \sin\phi - \overline{\tau}_w \mathscr{P}\, \Delta L = 0$$

[5]This section may be omitted without loss of continuity.

or
$$h_f = \frac{\Delta p}{\rho g} + \Delta z = \frac{\overline{\tau}_w}{\rho g}\frac{\Delta L}{A/\mathscr{P}} \tag{2.111}$$

This is identical to Eq. (2.52) except that (1) the shear stress is an average value integrated around the perimeter and (2) the length scale $A/\mathscr{P}$ takes the place of the pipe radius R. For this reason a noncircular duct is said to have a *hydraulic radius* R_h, defined by

$$R_h = \frac{A}{\mathscr{P}} = \frac{\text{cross-sectional area}}{\text{wetted perimeter}} \tag{2.112}$$

This concept receives constant use in open-channel flow , where the channel cross section is almost never circular. If, by comparison to Eq. (2.54) for pipe flow, we define the friction factor in terms of average shear

$$f_{\text{NCD}} = \frac{8\overline{\tau}_w}{\rho V^2} \tag{2.113}$$

where NCD stands for noncircular duct and $V = Q/A$ as usual, Eq. (2.111) becomes

$$h_f = f\frac{L}{4R_h}\frac{V^2}{2g} \tag{2.114}$$

This is equivalent to Eq. (2.55) for pipe flow except that d is replaced by $4R_h$. Therefore we customarily define the *hydraulic diameter* as

$$D_h = \frac{4A}{\mathscr{P}} = \frac{4 \times \text{area}}{\text{wetted perimeter}} = 4R_h \tag{2.115}$$

We should stress that the wetted perimeter includes all surfaces acted upon by the shear stress. For example, in a circular annulus, both the outer and the inner perimeter should be added. The fact that D_h equals $4R_h$ is just one of those things: Chalk it up to an engineer's sense of humor. Note that for the degenerate case of a circular pipe, $D_h = 4\pi R^2/(2\pi R) = 2R$, as expected.

We would therefore expect by dimensional analysis that this friction factor f, based upon hydraulic diameter as in Eq. (2.113), would correlate with the Reynolds number and roughness ratio based upon the hydraulic diameter

$$f = F\left(\frac{VD_h}{\nu}, \frac{\epsilon}{D_h}\right) \tag{2.116}$$

and this is the way the data are correlated. But we should not necessarily expect the Moody chart (Fig. 2.16) to hold exactly in terms of this new length scale. And it does not, but it is surprisingly accurate:

$$f \approx \begin{cases} \dfrac{64}{\mathrm{Re}_{D_h}} & \pm 40\% \quad \text{laminar flow} \\ f_{\text{Moody}}\left(\mathrm{Re}_{D_h}, \dfrac{\epsilon}{D_h}\right) & \pm 15\% \quad \text{turbulent flow} \end{cases} \tag{2.117}$$

Now let us look at some particular cases.

Flow between Parallel Plates (laminar or turbulent):

As shown in Fig. 2.17 , flow between parallel plates a distance h apart is the limiting case of flow through a very wide rectangular channel. For fully developed flow, $u = u(y)$ only, which satisfies continuity identically. The momentum equation in cartesian coordinates reduces to

$$0 = -\frac{dp}{dx} + \rho g_x + \frac{d\tau}{dy} \qquad \tau_{\text{lam}} = \mu\frac{du}{dy} \tag{2.118}$$

subject to no-slip conditions: $u = 0$ at $y = \pm h$. The laminar-flow solution was given as an example. Here we also allow for the possibility of a sloping channel, with a pressure gradient due to gravity. The solution is

$$u = \frac{1}{2\mu}\left[-\frac{d}{dx}(p + \rho g z)\right](h^2 - y^2) \tag{2.119}$$

If the channel has width b, the volume flow is

$$Q = \int_{-h}^{+h} u(y) b \, dy = \frac{bh^3}{3\mu}\left[-\frac{d}{dx}(p + \rho g z)\right]$$

or

$$V = \frac{Q}{bh} = \frac{h^2}{3\mu}\left[-\frac{d}{dx}(p + \rho g z)\right] = \frac{2}{3}u_{\max} \tag{2.120}$$

Note the difference between a parabola [Eq. 2.120] and a paraboloid [Eq. 2.68]: the average is two-thirds of the maximum velocity in plane flow and one-half in axisymmetric flow.

The wall shear stress in developed channel flow is a constant:

$$\tau_w = \mu\left|\frac{du}{dy}\right|_{y=+h} = h\left[-\frac{d}{dx}(p + \rho g z)\right] \tag{2.121}$$

This may be nondimensionalized as a friction factor:

$$f = \frac{8\tau_w}{\rho V^2} = \frac{24\mu}{\rho V h} = \frac{24}{\mathrm{Re}_h} \tag{2.122}$$

These are exact analytic laminar-flow results, so there is no reason to resort to the hydraulic-diameter concept. However, if we did use D_h, a discrepancy would arise. The hydraulic diameter of a wide channel is

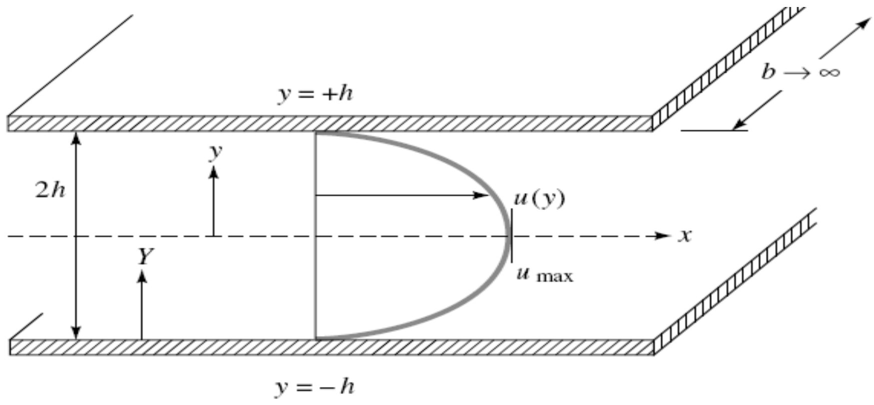


Fig. 2.17 Fully developed flow between parallel plates.

$$D_h = \frac{4A}{\rho} = \lim_{b\to\infty} \frac{4(2bh)}{2b + 4h} = 4h \tag{2.123}$$

or twice the distance between the plates. Substituting into Eq. (2.122), we obtain the interesting result

Parallel plates:
$$f_{\text{lam}} = \frac{96\mu}{\rho V(4h)} = \frac{96}{\text{Re}_{D_h}} \tag{2.124}$$

Thus, if we could not work out the laminar theory and chose to use the approximation $f \approx 64/\text{Re}_{D_h}$, we would be 33 percent low. The hydraulic-diameter approximation is relatively crude in laminar flow, as Eq. (2.117) states.

Just as in circular-pipe flow, the laminar solution above becomes unstable at about $\text{Re}_{D_h} \approx 2000$; transition occurs and turbulent flow results.

For turbulent flow between parallel plates, we can again use the logarithm law, eqs.2.84-88, as an approximation across the entire channel, using not y but a wall coordinate Y, as shown in Fig. 2.17 :

$$\frac{u(Y)}{u^*} \approx \frac{1}{\kappa}\ln\frac{Yu^*}{\nu} + B \qquad 0 < Y < h \tag{2.125}$$

This distribution looks very much like the flat turbulent profile for pipe flow in Fig. 2.14b, and the mean velocity is

$$V = \frac{1}{h}\int_0^h u\, dY = u^*\left(\frac{1}{\kappa}\ln\frac{hu^*}{\nu} + B - \frac{1}{\kappa}\right) \tag{2.126}$$

Recalling that $V/u^* = (8/f)^{1/2}$, we see that Eq. (2.126) is equivalent to a parallel-plate friction law. Rearranging and cleaning up the constant terms, we obtain

$$\frac{1}{f^{1/2}} \approx 2.0 \log\,(\text{Re}_{D_h} f^{1/2}) - 1.19 \tag{2.127}$$

where we have introduced the hydraulic diameter $D_h = 4h$. This is remarkably close to the pipe-friction law, Eq. (2.95). Therefore we conclude that the use of the hydraulic diameter in this turbulent case is quite successful. That turns out to be true for other noncircular turbulent flows also.

Equation (2.127) can be brought into exact agreement with the pipe law by rewriting it in the form

$$\frac{1}{f^{1/2}} = 2.0 \log\,(0.64\ \text{Re}_{D_h} f^{1/2}) - 0.8 \tag{2.128}$$

Thus the turbulent friction is predicted most accurately when we use an effective diameter D_{eff} equal to 0.64 times the hydraulic diameter. The effect on f itself is much less, about 10 percent at most. We can compare with Eq. (2.124) for laminar flow, which predicted

Parallel plates:
$$D_{\text{eff}} = \frac{64}{96}D_h = \frac{2}{3}D_h \tag{2.129}$$

This close resemblance ($0.64D_h$ versus $0.667D_h$) occurs so often in noncircular duct flow that we take it to be a general rule for computing turbulent friction in ducts:

$$D_{\text{eff}} = D_h = \frac{4A}{\mathcal{P}} \quad \text{reasonable accuracy}$$

$$D_{\text{eff}}(\text{laminar theory}) \quad \text{extreme accuracy} \tag{2.130}$$

Jones [10] shows that the effective-laminar-diameter idea collapses all data for rectangular ducts of arbitrary height-to-width ratio onto the Moody chart for pipe flow. We recommend this idea for all noncircular ducts.

Example 2.15:

Fluid flows at an average velocity of 6 ft/s between horizontal parallel plates a distance of 2.4 in apart. Find the head loss and pressure drop for each 100 ft of length for $\rho = 1.9$ slugs/ft^3 and (*a*) $\nu = 0.00002$ ft^3/s and (*b*) $\nu = 0.002$ ft^3/s. Assume smooth walls.

Solution : part a)

The viscosity $\mu = \rho\nu = 3.8 \times 10^{-5}$ slug/(ft · s). The spacing is $2h = 2.4$ in $= 0.2$ ft, and $D_h = 4h = 0.4$ ft. The Reynolds number is

$$\text{Re}_{D_h} = \frac{VD_h}{\nu} = \frac{(6.0 \text{ ft/s})(0.4 \text{ ft})}{0.00002 \text{ ft}^2/\text{s}} = 120{,}000$$

The flow is therefore turbulent. For reasonable accuracy, simply look on the Moody chart (Fig. 2.16) for smooth walls

$$f \approx 0.0173 \quad h_f \approx f\frac{L}{D_h}\frac{V^2}{2g} = 0.0173\,\frac{100}{0.4}\,\frac{(6.0)^2}{2(32.2)} \approx 2.42 \text{ ft} \qquad \textit{Ans. (a)}$$

Since there is no change in elevation,

$$\Delta p = \rho g h_f = 1.9(32.2)(2.42) = 148 \text{ lbf/ft}^2 \qquad \textit{Ans. (a)}$$

This is the head loss and pressure drop per 100 ft of channel. For more accuracy, take $D_{\text{eff}} = \frac{2}{3}D_h$ from laminar theory; then

$$\text{Re}_{\text{eff}} = \tfrac{2}{3}(120{,}000) = 80{,}000$$

and from the Moody chart read $f \approx 0.0189$ for smooth walls. Thus a better estimate is

$$h_f = 0.0189\,\frac{100}{0.4}\,\frac{(6.0)^2}{2(32.2)} = 2.64 \text{ ft}$$

and $$\Delta p = 1.9(32.2)(2.64) = 161 \text{ lbf/ft}^2 \qquad \textit{Better ans. (a)}$$

The more accurate formula predicts friction about 9 percent higher.

part b)

Compute $\mu = \rho\nu = 0.0038$ slug/(ft · s). The Reynolds number is $6.0(0.4)/0.002 = 1200$; therefore the flow is laminar, since Re is less than 2300.

You could use the laminar-flow friction factor, Eq. (2.124)

$$f_{\text{lam}} = \frac{96}{\text{Re}_{D_h}} = \frac{96}{1200} = 0.08$$

from which $$h_f = 0.08\,\frac{100}{0.4}\,\frac{(6.0)^2}{2(32.2)} = 11.2 \text{ ft}$$

and $$\Delta p = 1.9(32.2)(11.2) = 684 \text{ lbf/ft}^2 \qquad \textit{Ans. (b)}$$

Alternately you can finesse the Reynolds number and go directly to the appropriate laminar-flow formula, Eq. (2.120)

$$V = \frac{h^2}{3\mu}\,\frac{\Delta p}{L}$$

or $$\Delta p = \frac{3(6.0 \text{ ft/s})[0.0038 \text{ slug/(ft} \cdot \text{s)}](100 \text{ ft})}{(0.1 \text{ ft})^2} = 684 \text{ slugs/(ft} \cdot \text{s}^2) = 684 \text{ lbf/ft}^2$$

and $$h_f = \frac{\Delta p}{\rho g} = \frac{684}{1.9(32.2)} = 11.2 \text{ ft}$$

This is one of those—perhaps unexpected—problems where the laminar friction is greater than the turbulent friction.

Flow through a Concentric Annuls:

Consider steady axial laminar flow in the annular space between two concentric cylinders, as in Fig. 2.18 . There is no slip at the inner ($r = b$) and outer radius ($r = a$). For $u = u(r)$ only, the governing relation is Eq. (2.59)

$$\frac{d}{dr}\left(r\mu\frac{du}{dr}\right) = Kr \qquad K = \frac{d}{dx}(p + \rho g z) \tag{2.131}$$

Integrate this twice

$$u = \frac{1}{4}r^2\frac{K}{\mu} + C_1 \ln r + C_2$$

The constants are found from the two no-slip conditions

$$u(r = a) = 0 = \frac{1}{4}a^2\frac{K}{\mu} + C_1 \ln a + C_2$$

$$u(r = b) = 0 = \frac{1}{4}b^2\frac{K}{\mu} + C_1 \ln b + C_2$$

The final solution for the velocity profile is

$$u = \frac{1}{4\mu}\left[-\frac{d}{dx}(p + \rho g z)\right]\left[a^2 - r^2 + \frac{a^2 - b^2}{\ln (b/a)}\ln\frac{a}{r}\right] \tag{2.132}$$

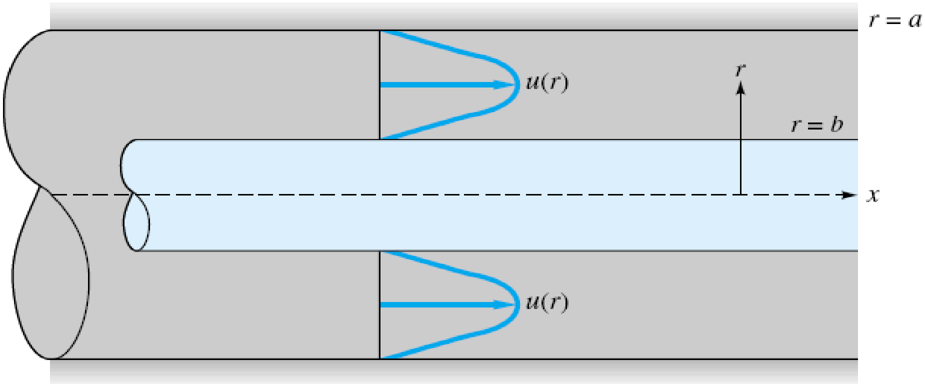


Fig. 2.18 Fully developed flow through a concentric annulus.

The volume flow is given by

$$Q = \int_b^a u2\pi r\, dr = \frac{\pi}{8\mu}\left[-\frac{d}{dx}(p + \rho g z)\right]\left[a^4 - b^4 - \frac{(a^2 - b^2)^2}{\ln (a/b)}\right] \tag{2.133}$$

The velocity profile $u(r)$ resembles a parabola wrapped around in a circle to form a split doughnut, as in Fig. 2.18 . The maximum velocity occurs at the radius

$$r' = \left[\frac{a^2 - b^2}{2\ln (a/b)}\right]^{1/2} \qquad u = u_{max} \tag{2.134}$$

This maximum is closer to the inner radius but approaches the midpoint between cylinders as the clearance $a - b$ becomes small. Some numerical values are as follows:

$\frac{b}{a}$	0.01	0.1	0.2	0.5	0.8	0.9	0.99
$\frac{r' - b}{a - b}$	0.323	0.404	0.433	0.471	0.491	0.496	0.499

Also, as the clearance becomes small, the profile approaches a parabolic distribution, as if the flow were between two parallel plates.

It is confusing to base the friction factor on the wall shear because there are two shear stresses, the inner stress being greater than the outer. It is better to define f with respect to the head loss, as in Eq. (2.114),

$$f = h_f \frac{D_h}{L} \frac{2g}{V^2} \qquad \text{where } V = \frac{Q}{\pi(a^2 - b^2)} \tag{2.135}$$

The hydraulic diameter for an annulus is

$$D_h = \frac{4\pi(a^2 - b^2)}{2\pi(a + b)} = 2(a - b) \tag{2.136}$$

It is twice the clearance, rather like the parallel-plate result of twice the distance between plates [Eq. (2.123)].

Substituting h_f, D_h, and V into Eq. (2.135), we find that the friction factor for laminar flow in a concentric annulus is of the form

$$f = \frac{64\zeta}{\mathrm{Re}_{D_h}} \qquad \zeta = \frac{(a - b)^2(a^2 - b^2)}{a^4 - b^4 - (a^2 - b^2)^2/\ln(a/b)} \tag{2.137}$$

The dimensionless term ζ is a sort of correction factor for the hydraulic diameter. We could rewrite Eq. (2.137) as

$$\text{Concentric annulus:} \qquad f = \frac{64}{\mathrm{Re}_{\mathrm{eff}}} \qquad \mathrm{Re}_{\mathrm{eff}} = \frac{1}{\zeta}\mathrm{Re}_{D_h} \tag{2.138}$$

Some numerical values of $f\,\mathrm{Re}_{D_h}$ and $D_{\mathrm{eff}}/D_h = 1/\zeta$ are given in Table 2.3 .

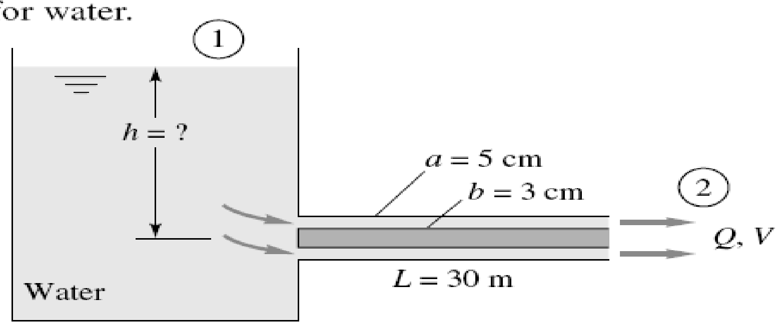
For turbulent flow through a concentric annulus, the analysis might proceed by patching together two logarithmic-law profiles, one going out from the inner wall to meet the other coming in from the outer wall. We omit such a scheme here and proceed directly to the friction factor. According to the general rule proposed in Eq. (2.130), turbulent friction is predicted with excellent accuracy by replacing d in the Moody chart by $D_{\mathrm{eff}} = 2(a - b)/\zeta$, with values listed in Table 2.3 .[6] This idea includes roughness also (replace ϵ/d in the chart by $\epsilon/D_{\mathrm{eff}}$). For a quick design number with about 10 percent accuracy, one can simply use the hydraulic diameter $D_h = 2(a - b)$.

Table 2.3 Laminar Friction Factors for a Concentric Annulus

b/a	$f\,\mathrm{Re}_{D_h}$	$D_{\mathrm{eff}}/D_h = 1/\zeta$
0.0	64.0	1.000
0.00001	70.09	0.913
0.0001	71.78	0.892
0.001	74.68	0.857
0.01	80.11	0.799
0.05	86.27	0.742
0.1	89.37	0.716
0.2	92.35	0.693
0.4	94.71	0.676
0.6	95.59	0.670
0.8	95.92	0.667
1.0	96.0	0.667

Example 2.16:

What should the reservoir level h be to maintain a flow of 0.01 m^3/s through the commercial steel annulus 30 m long shown in Fig. E2.16? Neglect entrance effects and take $\rho = 1000$ kg/m^3 and $\nu = 1.02 \times 10^{-6}$ m^2/s for water.



E2.16

Solution

Compute the average velocity and hydraulic diameter

$$V = \frac{Q}{A} = \frac{0.01\ \mathrm{m^3/s}}{\pi[(0.05\ \mathrm{m})^2 - (0.03\ \mathrm{m})^2]} = 1.99\ \mathrm{m/s}$$

$$D_h = 2(a - b) = 2(0.05 - 0.03)\ \mathrm{m} = 0.04\ \mathrm{m}$$

Apply the steady-flow energy equation between sections 1 and 2:

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \left(\frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2\right) + gh_f$$

But $p_1 = p_2 = p_a$, $V_1 \approx 0$, and $V_2 = V$ in the pipe. Therefore solve for

$$h_f = f\frac{L}{D_h}\frac{V^2}{2g} = z_1 - z_2 - \frac{V^2}{2g}$$

But $z_1 - z_2 = h$, the desired reservoir height. Thus, finally,

$$h = \frac{V^2}{2g}\left(1 + f\frac{L}{D_h}\right) \tag{1}$$

Since V, L, and D_h are known, our only remaining problem is to compute the annulus friction factor f. For a quick approximation, take $D_{\text{eff}} = D_h = 0.04$ m. Then

$$\text{Re}_{D_h} = \frac{VD_h}{\nu} = \frac{1.99(0.04)}{1.02 \times 10^{-6}} = 78{,}000$$

$$\frac{\epsilon}{D_h} = \frac{0.046 \text{ mm}}{40 \text{ mm}} = 0.00115$$

[6]Jones and Leung [44] show that data for annular flow also satisfy the effective-laminar-diameter idea.

where $\epsilon = 0.046$ mm has been read from Table 2.1 for commercial steel surfaces. From the Moody chart, read $f = 0.0232$. Then, from Eq. (1) above,

$$h \approx \frac{(1.99 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)}\left(1 + 0.0232\frac{30 \text{ m}}{0.04 \text{ m}}\right) = 3.71 \text{ m} \qquad \textit{Crude ans.}$$

For better accuracy, take $D_{\text{eff}} = D_h/\zeta = 0.670D_h = 2.68$ cm, where the correction factor 0.670 has been read from Table 6.3 for $b/a = \frac{3}{5} = 0.6$. Then the corrected Reynolds number and roughness ratio are

$$\text{Re}_{\text{eff}} = \frac{VD_{\text{eff}}}{\nu} = 52{,}300 \qquad \frac{\epsilon}{D_{\text{eff}}} = 0.00172$$

From the Moody chart, read $f = 0.0257$. Then the improved computation for reservoir height is

$$h = \frac{(1.99 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)}\left(1 + 0.0257\frac{30 \text{ m}}{0.04 \text{ m}}\right) = 4.09 \text{ m} \qquad \textit{Better ans.}$$

The uncorrected hydraulic-diameter estimate is about 9 percent low. Note that we do *not* replace D_h by D_{eff} in the ratio L/D_h in Eq. (1) since this is implicit in the definition of friction factor.

Flow in Other Noncircular Cross-Sections:

In principle, any duct cross section can be solved analytically for the laminar-flow velocity distribution, volume flow, and friction factor. This is because any cross section can be mapped onto a circle by the methods of complex variables, and other powerful analytical techniques are also available. Many examples are given by White [3, pp. 119–122], Berker [11], and Olson and Wright [12, pp. 315–317]. Reference 34 is devoted entirely to laminar duct flow.

In general, however, most unusual duct sections have strictly academic and not commercial value. We list here only the rectangular and isosceles-triangular sections, in Table 6.4, leaving other cross sections for you to find in the references.

For turbulent flow in a duct of unusual cross section, one should replace d by D_h on the Moody chart if no laminar theory is available. If laminar results are known, such as Table 2.4 , replace d by $D_{\text{eff}} = [64/(f\,\text{Re})]D_h$ for the particular geometry of the duct.

For laminar flow in rectangles and triangles, the wall friction varies greatly, being largest near the midpoints of the sides and zero in the corners. In turbulent flow through the same sections, the shear is nearly constant along the sides, dropping off sharply to zero in the corners. This is because of the phenomenon of turbulent *secondary flow*, in which there are nonzero mean velocities v and w in the plane of the cross section. Some measurements of axial velocity and secondary-flow patterns are shown in Fig. 2.19 , as sketched by Nikuradse in his 1926 dissertation. The secondary-flow "cells" drive the mean flow toward the corners, so that the axial-velocity contours are similar to the cross section and the wall shear is nearly constant. This is why the hydraulic-diameter concept is so successful for turbulent flow. Laminar flow in a straight noncircular duct has no secondary flow. An accurate theoretical prediction of turbulent secondary flow has yet to be achieved, although numerical models are improving [36].

Table 2.4 Laminar Friction Constants fRe for Rectangular and Triangular Ducts

Rectangular		Isosceles triangle	
b/a	$f\text{Re}_{D_h}$	θ, deg	$f\text{Re}_{D_h}$
0.0	96.00	0	48.0
0.05	89.91	10	51.6
0.1	84.68	20	52.9
0.125	82.34	30	53.3
0.167	78.81	40	52.9
0.25	72.93	50	52.0
0.4	65.47	60	51.1
0.5	62.19	70	49.5
0.75	57.89	80	48.3
1.0	56.91	90	48.0

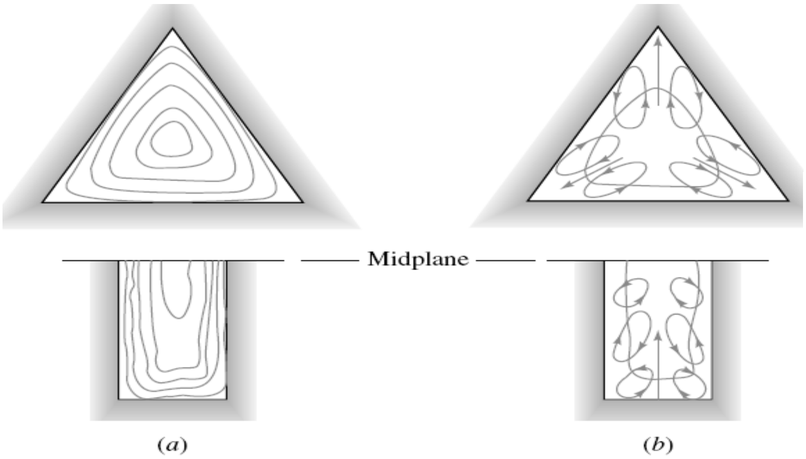


Fig. 2.19 Illustration of secondary turbulent flow in noncircular ducts: (*a*) axial mean-velocity contours; (*b*) secondary-flow cellular motions. *(After J. Nikuradse, dissertation, Göttingen, 1926.)*

__Example 2.17:__

Air, with $\rho = 0.00237$ slug/ft^3 and $\nu = 0.000157$ ft^2/s, is forced through a horizontal square 9-by 9-in duct 100 ft long at 25 ft^3/s. Find the pressure drop if $\epsilon = 0.0003$ ft.

Solution

Compute the mean velocity and hydraulic diameter

$$V = \frac{25 \text{ ft}^3/\text{s}}{(0.75 \text{ ft})^2} = 44.4 \text{ ft/s}$$

$$D_h = \frac{4A}{\mathcal{P}} = \frac{4(81 \text{ in}^2)}{36 \text{ in}} = 9 \text{ in} = 0.75 \text{ ft}$$

From Table 2.4 , for $b/a = 1.0$, the effective diameter is

$$D_{\text{eff}} = \frac{64}{56.91} D_h = 0.843 \text{ ft}$$

whence

$$\text{Re}_{\text{eff}} = \frac{VD_{\text{eff}}}{\nu} = \frac{44.4(0.843)}{0.000157} = 239{,}000$$

$$\frac{\epsilon}{D_{\text{eff}}} = \frac{0.0003}{0.843} = 0.000356$$

From the Moody chart, read $f = 0.0177$. Then the pressure drop is

$$\Delta p = \rho g h_f = \rho g \left(f \frac{L}{D_h} \frac{V^2}{2g} \right) = 0.00237(32.2)\left[0.0177 \frac{100}{0.75} \frac{44.4^2}{2(32.2)}\right]$$

or

$$\Delta p = 5.5 \text{ lbf/ft}^2 \qquad \textit{Ans.}$$

Pressure drop in air ducts is usually small because of the low density.

Oral Questions for the ***Viscous Flow Parts (1 & 2)***

Note: Questions from 1-6 are in Fluid Report # (1) in Part-1

7- What do we mean (both physically and mathematically) by a fully-developed flow? Give an example describing the fully-developed velocity field in a circular pipe. How do we get the equation for this velocity field and also for the average velocity and for the shear stress distribution in this field.

8- Prove that the time-derivative operator (called total or substantial derivative) following a fluid particle is: $d/dt = \partial / \partial t + (\underline{\mathbf{V}} \cdot \underline{\blacktriangledown})$, where $\underline{\blacktriangledown}$ is the gradient operator.

9- What do you know about the conservation equations in Fluid Mechanics? Using the differential analysis method, stat and discuss two of the main conservation equations of fluid mechanics. Show all the non-linear terms in those equations. What is the divergence of the velocity vector field?. Can we write the momentum equations for a Non-Newtonian fluid? How?.

10- What do we mean (both physically and mathematically) by a fully-developed flow? Give an example describing the fully-developed velocity field between two parallel fixed plates separated by a small distance, h, with zero pressure drop in the direction of the flow. How do we get the equation for this velocity field and also for the average velocity and for the shear stress distribution in this field.

11- What do we mean (both physically and mathematically) by a fully-developed flow? Give an example describing the fully-developed velocity field between two parallel plates separated by a small distance, h, with the upper plate moving with a velocity U_o in the +ve x-direction and with zero pressure drop in the direction of the flow. How do we get the equation for this velocity field and also for the average velocity and for the shear stress distribution in this field.

12- What do we mean (both physically and mathematically) by a fully-developed flow? Give an example describing the fully-developed velocity field between two parallel plates separated by a small distance, h, with the upper plate moving with a velocity U_o in the -ve x-direction and with zero pressure drop in the direction of the flow. How do we get the equation for this velocity field and also for the average velocity and for the shear stress distribution in this field.

13- Using the cartesian coordinates, write down and discuss the meaning of each term and show all the differences you know between the Navier-Stoke's equations and the Euler's equations. Can we use Euler's equations to solve the flow in long pipes? Why? What is the relation between Euler's equations and Bernolli's equation?

14-The axial velocity profile, u, in incompressible laminar and turbulent flow in a circular pipe may be well approximated by : a) $u = U_{Lmax} (1- r^2/R^2)$ and b) $u = U_{Tmax} (1- r/R)^{1/7}$ where R is the pipe radius. Find the volume flow rate, the mean velocity, shear stress at the wall and the friction force on the pipe wall if the pipe length is L.

15- A tank of volume V contains a liquid of an initial density ρ_i and a second liquid of a density denoted by ρ_{in} enters the tank steadily with a mass flow rate min and mixes thoroughly with the fluid in the tank. The liquid level in the tank is kept constant by allowing m_{out} flow out of the side of the tank. Drive an expression for the time rate of change of the density $\rho(t)$ of the liquid in the tank, and the time required for the density in the tank to reach the value of ρ_f .

16- The diameter of a pipe bend is 300 mm at inlet and 150 mm at outlet, the flow is turned 120^o in a vertical plane. The axis at inlet is horizontal and the center of the outlet section is 1.4m below the center of inlet section. The total volume of fluid contained in the bend is $0.085 m^3$. Neglecting friction, find the magnitude and direction of the net force exerted on the bend by water flowing through it at $0.23 m^3/sec$ if the inlet pressure is 140 kPa.

17- The flow rate is $0.25 m^3/s$ into a convergent nozzle of 0.6 m height at entrance and 0.3m height at exit. Find the velocity, acceleration and pressure fields through the nozzle. (assume the nozzle length = 1.5 m)

18- The flow rate is $0.25 m^3/s$ into a divergent nozzle of 0.3 m height at entrance and 0.6m height at exit. Find the velocity, acceleration and pressure fields through the nozzle. (assume the nozzle length = 1.5 m)

19- Which of the following motions are kinematically possible for incompressible flow (k and Q are constants): i) u = k x , v =k y, w =-2k z ii) V_r =- Q/2Л r , V_θ k/2Л r iii) V_r = k cos θ, V_θ = -k sin θ

20- For a 2-D flow field in the xy-plane, the y component of the velocity is given by: $v = y^2 - 2x + 2y$. Determine a possible x-component for a steady incompressible flow. Is it also valid for unsteady flow? How many possible x-components are there? Why?.

21- For a 2-D flow field in the xy-plane, the x component of the velocity is given by: $u = y^2 - 2x + 2y$. Determine a possible y-component for a steady incompressible flow. Is it also valid for unsteady flow? How many possible x-components are there? Why?.

22- Prove that the equation of continuity for 2-D incompressible flow in polar coordinates is in the form: $\partial V_r/\partial r + V_r/r + 1/r\,(\partial V_\theta/\partial\theta) = 0$

23- Starting from the Navier-Stokes equations, drive the well known Bernoulli's equation. State all the assumptions made.

24- Explain the physical meaning and the mathematical equations for both the divergence operator and the curl operator as applied on a vector field. Take the velocity field **V** as an example. (hint: prove that the divergence of **V** = the rate of volume expansion of fluid element per unit initial volume) . Prove also that for an incompressible velocity field, the divergence of **V** = 0. What is the relationship between the curle of **V** and the rotation in the velocity field.

25- Prove that the axial velocity profile in a laminar flow in a tube of radius R is: $u(r) = (- R^2/4\mu)\,(dp/dx)\,[\,1 - (r/R)^2\,]$,where x is along the centerline of the tube.
How can a simple and an accurate viscosity meter be made using equations of laminar flow in a pipe. The viscosity of fluid passing through a length of a thin tube can be calculated if the volumetric flow rate and pressure drop are measured and the tube geometry is known?.
A test on such viscosity meter gave the following data: Q = 880 mm^3/sec. , tube diameter, d = 0.5mm , tube length, L = 1.0 m, the pressure drop, Δp = 1 Mpa. If the specific gravity =1.0, Find the dynamic viscosity, μ of the fluid in the tube (is the flow realy laminar or not ? check on that assumption).

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Word Problems on Part (2):

1 In fully developed straight-duct flow, the velocity profiles do not change (why?), but the pressure drops along the pipe axis. Thus there is pressure work done on the fluid. If, say, the pipe is insulated from heat loss, where does this energy go? Make a thermodynamic analysis of the pipe flow.

2 From the Moody chart (Fig. 2.16), rough surfaces, such as sand grains or ragged machining, do not affect laminar flow. Can you explain why? They *do* affect turbulent flow. Can you develop, or suggest, an analytical-physical model of turbulent flow near a rough surface which might be used to predict the known increase in pressure drop?

3 Differentiation of the laminar pipe-flow solution, Eq. (2.65), shows that the fluid shear stress $\tau(r)$ varies linearly from zero at the axis to τ_w at the wall. It is claimed that this is also true, at least in the time mean, for fully developed *turbulent* flow. Can you verify this claim analytically?

4 A porous medium consists of many tiny tortuous passages, and Reynolds numbers based on pore size are usually very low, of order unity. In 1856 H. Darcy proposed that the pressure gradient in a porous medium was directly proportional to the volume-averaged velocity **V** of the fluid:

$$\nabla p = -\frac{\mu}{K}\mathbf{V}$$

where K is termed the *permeability* of the medium. This is now called *Darcy's law* of porous flow. Can you make a Poiseuille flow model of porous-media flow which verifies Darcy's law? Meanwhile, as the Reynolds number increases, so that $VK^{1/2}/\nu > 1$, the pressure drop becomes nonlinear, as was shown experimentally by P. H. Forscheimer as early as 1782. The flow is still decidedly laminar, yet the pressure gradient is quadratic:

$$\nabla p = -\frac{\mu}{K}\mathbf{V} - C|V|\mathbf{V} \quad \text{Darcy-Forscheimer law}$$

where C is an empirical constant. Can you explain the reason for this nonlinear behavior?

5 One flowmeter device, in wide use in the water supply and gasoline distribution industries, is the *nutating disk*. Look this up in the library, and explain in a brief report how it works and the advantages and disadvantages of typical designs.

Problems on parts (1) & (2):

1 The velocity in a certain two-dimensional flow field is given by the equation

$$\mathbf{V} = 2xt\hat{\mathbf{i}} - 2yt\hat{\mathbf{j}}$$

where the velocity is in ft/s when x, y, and t are in feet and seconds, respectively. Determine expressions for the local and convective components of acceleration in the x and y directions. What is the magnitude and direction of the velocity and the acceleration at the point $x = y = 2$ ft at the time $t = 0$?

2 Repeat Problem 1 if the flow field is described by the equation

$$\mathbf{V} = 3(x^2 - y^2)\hat{\mathbf{i}} - 6xy\hat{\mathbf{j}}$$

where the velocity is in ft/s when x and y are in feet.

3 The velocity in a certain flow field is given by the equation

$$\mathbf{V} = x\hat{\mathbf{i}} + x^2z\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$$

Determine the expressions for the three rectangular components of acceleration.

4 The three components of velocity in a flow field are given by

$$u = x^2 + y^2 + z^2$$
$$v = xy + yz + z^2$$
$$w = -3xz - z^2/2 + 4$$

(a) Determine the volumetric dilatation rate and interpret the results. **(b)** Determine an expression for the rotation vector. Is this an irrotational flow field?

5 Determine an expression for the vorticity of the flow field described by

$$\mathbf{V} = -xy^3\hat{\mathbf{i}} + y^4\hat{\mathbf{j}}$$

Is the flow irrotational?

6 A one-dimensional flow is described by the velocity field

$$u = ay + by^2$$
$$v = w = 0$$

where a and b are constants. Is the flow irrotational? For what combination of constants (if any) will the rate of angular deformation as given by Eq. 6.18 be zero?

7 For incompressible fluids the volumetric dilatation rate must be zero; that is, $\nabla \cdot \mathbf{V} = 0$. For what combination of constants, a, b, c, and e can the velocity components

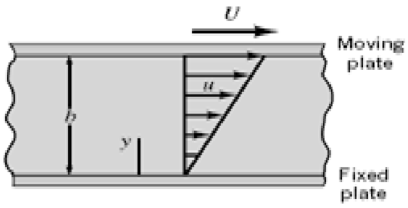
$$u = ax + by$$
$$v = cx + ey$$
$$w = 0$$

be used to describe an incompressible flow field?

8 An incompressible viscous fluid is placed between two large parallel plates as shown in Fig. P.8. The bottom plate is fixed and the upper plate moves with a constant velocity, U. For these conditions the velocity distribution between the plates is linear and can be expressed as

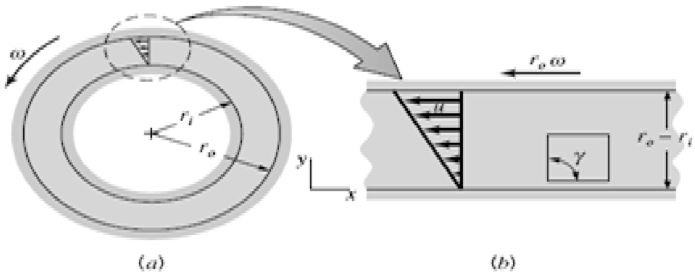
$$u = U\frac{y}{b}$$

Determine: **(a)** the volumetric dilatation rate, **(b)** the rotation vector, **(c)** the vorticity, and **(d)** the rate of angular deformation.



■ FIGURE P8

9 A viscous fluid is contained in the space between concentric cylinders. The inner wall is fixed, and the outer wall rotates with an angular velocity ω. (See Fig. P 9*a* and Video V6.1.) Assume that the velocity distribution in the gap is linear as illustrated in Fig. P 9*b*. For the small rectangular element shown in Fig. P 9*b*, determine the rate of change of the right angle γ due to the fluid motion. Express your answer in terms of r_0, r_i, and ω.



■ FIGURE P9

10 Some velocity measurements in a three-dimensional incompressible flow field indicate that $u = 6xy^2$ and $v = -4y^2z$. There is some conflicting data for the velocity component in the z direction. One set of data indicates that $w = 4yz^2$, and the other set indicates that $w = 4yz^2 - 6y^2z$. Which set do you think is correct? Explain.

11 The velocity components of an incompressible, two-dimensional velocity field are given by the equations

$$u = 2xy$$
$$v = x^2 - y^2$$

Show that the flow is irrotational and satisfies conservation of mass.

12 For each of the following stream functions, with units of m²/s, determine the magnitude and the angle the velocity vector makes with the x-axis at $x = 1$ m, $y = 2$ m. Locate any stagnation points in the flow field.
(a) $\psi = xy$
(b) $\psi = -2x^2 + y$

13 The stream function for a certain incompressible flow field is

$$\psi = 10y + e^{-y} \sin x$$

Is this an irrotational flow field? Justify your answer with the necessary calculations.

14 The stream function for an incompressible, two-dimensional flow field is

$$\psi = ay^2 - bx$$

where a and b are constants. Is this an irrotational flow? Explain.

15 The velocity components for an incompressible, plane flow are

$$v_r = Ar^{-1} + Br^{-2} \cos\theta$$
$$v_\theta = Br^{-2} \sin\theta$$

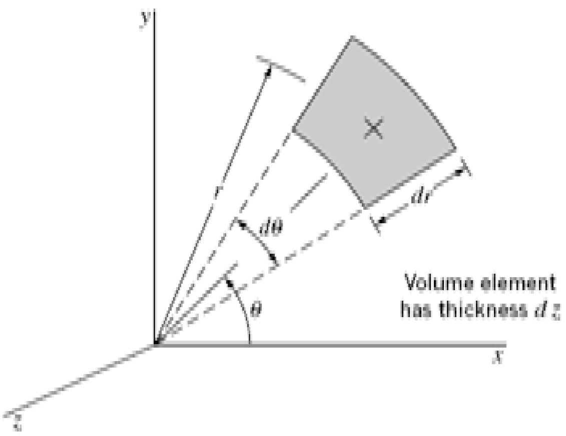
where A and B are constants. Determine the corresponding stream function.

16 For a certain two-dimensional flow field

$$u = 0$$
$$v = V$$

(a) What are the corresponding radial and tangential velocity components? (b) Determine the corresponding stream function expressed in Cartesian coordinates and in cylindrical polar coordinates.

17 Make use of the control volume shown in Fig. P17 to derive the continuity equation in cylindrical coordinates (Eq. 2.6 in text).



■ FIGURE P17

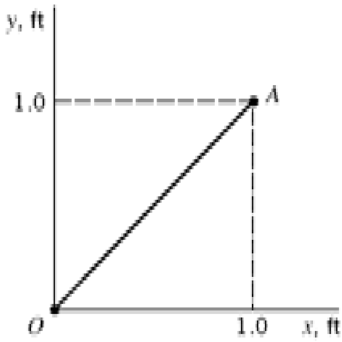
18 It is proposed that a two-dimensional, incompressible flow field be described by the velocity components

$$u = Ay$$
$$v = Bx$$

where A and B are both positive constants. (a) Will the continuity equation be satisfied? (b) Is the flow irrotational? (c) Determine the equation for the streamlines and show a sketch of the streamline that passes through the origin. Indicate the direction of flow along this streamline.

19 In a certain steady, two-dimensional flow field the fluid density varies linearly with respect to the coordinate x; that is, $\rho = Ax$ where A is a constant. If the x component of velocity u is given by the equation $u = y$, determine an expression for v.

20 In a two-dimensional, incompressible flow field, the x component of velocity is given by the equation $u = 2x$. (a) Determine the corresponding equation for the y component of velocity if $v = 0$ along the x axis. (b) For this flow field, what is the magnitude of the average velocity of the fluid crossing the surface OA of Fig. P 20? Assume that the velocities are in feet per second when x and y are in feet.



■ FIGURE P20

21 The radial velocity component in an incompressible, two-dimensional flow field ($v_z = 0$) is

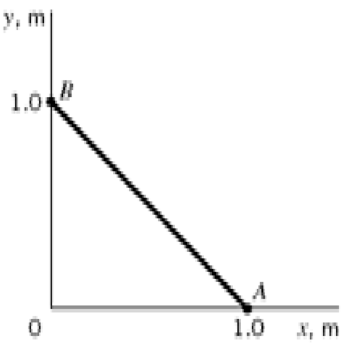
$$v_r = 2r + 3r^2 \sin\theta$$

Determine the corresponding tangential velocity component, v_θ, required to satisfy conservation of mass.

22 The stream function for an incompressible flow field is given by the equation

$$\psi = 3x^2y - y^3$$

where the stream function has the units of m^2/s with x and y in meters. (a) Sketch the streamline(s) passing through the origin. (b) Determine the rate of flow across the straight path AB shown in Fig. P 22.



■ FIGURE P22

23 The streamlines in a certain incompressible, two-dimensional flow field are all concentric circles so that $v_r = 0$. Determine the stream function for (a) $v_\theta = Ar$ and for (b) $v_\theta = Ar^{-1}$, where A is a constant.

71 For a two-dimensional incompressible flow in the x–y plane show that the z component of the vorticity, ζ_z, varies in accordance with the equation

$$\frac{D\zeta_z}{Dt} = \nu \nabla^2 \zeta_z$$

What is the physical interpretation of this equation for a non-viscous fluid? *Hint*: This *vorticity transport equation* can be derived from the Navier–Stokes equations by differentiating and eliminating the pressure between Eqs. **2.9 a** and **2.9 b**

72 The velocity of a fluid particle moving along a horizontal streamline that coincides with the x axis in a plane, two-dimensional, incompressible flow field was experimentally found to be described by the equation $u = x^2$. Along this streamline determine an expression for **(a)** the rate of change of the v component of velocity with respect to y, **(b)** the acceleration of the particle, and **(c)** the pressure gradient in the x direction. The fluid is Newtonian.

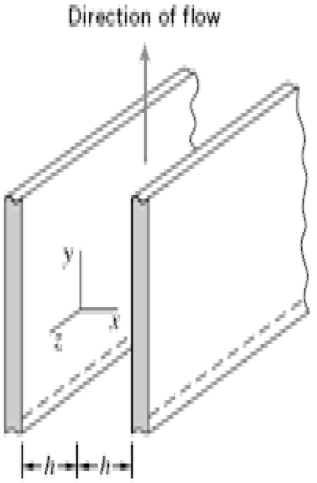
73 Two horizontal, infinite, parallel plates are spaced a distance b apart. A viscous liquid is contained between the plates. The bottom plate is fixed, and the upper plate moves parallel to the bottom plate with a velocity U. Because of the no-slip boundary condition (see Video V6.5), the liquid motion is caused by the liquid being dragged along by the moving boundary. There is no pressure gradient in the direction of flow. Note that this is a so-called simple *Couette flow* discussed in Section . **(a)** Start with the Navier–Stokes equations and determine the velocity distribution between the plates. **(b)** Determine an expression for the flowrate passing between the plates (for a unit width). Express your answer in terms of b and U.

74 Oil (SAE 30) at 15.6 °C flows steadily between fixed, horizontal, parallel plates. The pressure drop per unit length along the channel is 20 kPa/m, and the distance between the plates is 4mm. The flow is laminar. Determine: **(a)** the volume rate of flow (per meter of width), **(b)** the magnitude and direction of the shearing stress acting on the bottom plate, and **(c)** the velocity along the centerline of the channel.

75 Two fixed, horizontal, parallel plates are spaced 0.2 in. apart. A viscous liquid ($\mu = 8 \times 10^{-3}$ lb · s/ft^2, $SG = 0.9$) flows between the plates with a mean velocity of 0.7 ft/s. Determine the pressure drop per unit length in the direction of flow. What is the maximum velocity in the channel?

76 A layer of viscous liquid of constant thickness (no velocity perpendicular to plate) flows steadily down an infinite, inclined plane. Determine, by means of the Navier–Stokes equations, the relationship between the thickness of the layer and the discharge per unit width. The flow is laminar, and assume air resistance is negligible so that the shearing stress at the free surface is zero.

77 A viscous, incompressible fluid flows between the two infinite, vertical, parallel plates of Fig. P6.77. Determine, by use of the Navier–Stokes equations, an expression for the pressure gradient in the direction of flow. Express your answer in terms of the mean velocity. Assume that the flow is laminar, steady, and uniform.

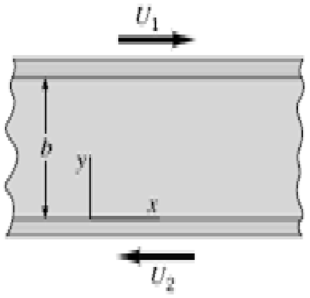


■ FIGURE P 77

78 A fluid of density ρ flows steadily *downward* between the two vertical, infinite, parallel plates shown in the figure for Problem 6.77. The flow is fully developed and laminar. Make use of the Navier–Stokes equation to determine the relationship between the discharge and the other parameters involved, for the case in which the change in pressure along the channel is zero.

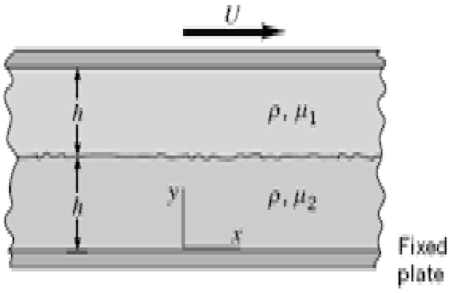
79 Due to the no-slip condition, as a solid is pulled out of a viscous liquid some of the liquid is also pulled along as described in Example 6.9 and shown in Video V6.5. Based on the results given in Example 6.9, show on a dimensionless plot the velocity distribution in the fluid film (v/V_0 vs. x/h) when the average film velocity, V, is 10% of the belt velocity, V_0.

80 An incompressible, viscous fluid is placed between horizontal, infinite, parallel plates as is shown in Fig. P6.80. The two plates move in opposite directions with constant velocities, U_1 and U_2, as shown. The pressure gradient in the x direction is zero, and the only body force is due to the fluid weight. Use the Navier–Stokes equations to derive an expression for the velocity distribution between the plates. Assume laminar flow.



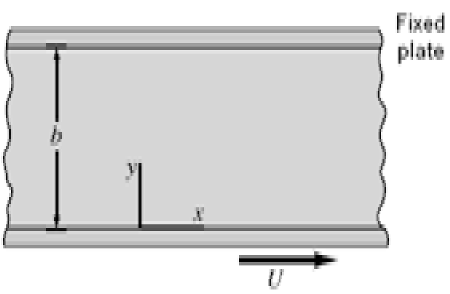
■ FIGURE P 80

81 Two immiscible, incompressible, viscous fluids having the same densities but different viscosities are contained between two infinite, horizontal, parallel plates (Fig. P 81). The bottom plate is fixed and the upper plate moves with a constant velocity U. Determine the velocity at the interface. Express your answer in terms of U, μ_1, and μ_2. The motion of the fluid is caused entirely by the movement of the upper plate; that is, there is no pressure gradient in the x direction. The fluid velocity and shearing stress are continuous across the interface between the two fluids. Assume laminar flow.



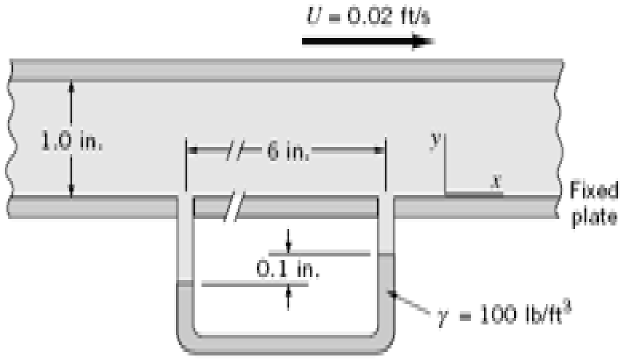
■ FIGURE P 81

82 The viscous, incompressible flow between the parallel plates shown in Fig. P 82 is caused by both the motion of the bottom plate and a pressure gradient, $\partial p/\partial x$. As noted in Section **2.1.2**, an important dimensionless parameter for this type of problem is $P = -(b^2/2\,\mu U)(\partial p/\partial x)$ where μ is the fluid viscosity. Make a plot of the dimensionless velocity distribution (similar to that shown in Fig. **2.31 b**) for $P = 3$. For this case where does the maximum velocity occur?



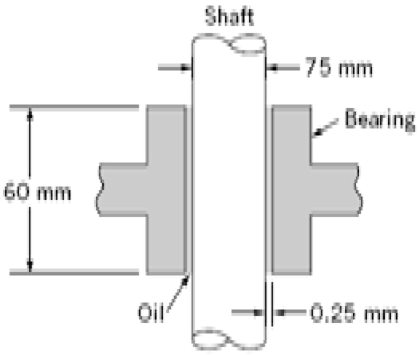
■ FIGURE P 82

83 A viscous fluid (specific weight = 80 lb/ft^3; viscosity = 0.03 lb · s/ft^2) is contained between two infinite, horizontal parallel plates as shown in Fig. P 83. The fluid moves between the plates under the action of a pressure gradient, and the upper plate moves with a velocity U while the bottom plate is fixed. A U-tube manometer connected between two points along the bottom indicates a differential reading of 0.1 in. If the upper plate moves with a velocity of 0.02 ft/s, at what distance from the bottom plate does the maximum velocity in the gap between the two plates occur? Assume laminar flow.



■ FIGURE P 83

84 A vertical shaft passes through a bearing and is lubricated with an oil having a viscosity of 0.2 N · s/m^2 as shown in Fig. P 84. Assume that the flow characteristics in the gap between the shaft and bearing are the same as those for laminar flow between infinite parallel plates with zero pressure gradient in the direction of flow. Estimate the torque required to overcome viscous resistance when the shaft is turning at 80 rev/min.



■ FIGURE P 84

85 A viscous fluid is contained between two long concentric cylinders. The geometry of the system is such that the flow between the cylinders is approximately the same as the laminar flow between two infinite parallel plates. **(a)** Determine an expression for the torque required to rotate the outer cylinder with an angular velocity ω. The inner cylinder is fixed. Express your answer in terms of the geometry of the system, the viscosity of the fluid, and the angular velocity. **(b)** For a small, rectangular element located at the fixed wall determine an expression for the rate of angular deformation of this element. (See Video V6.1 and Fig. P 9.)

86 Oil (SAE 30) flows between parallel plates spaced 5 mm apart. The bottom plate is fixed, but the upper plate moves with a velocity of 0.2 m/s in the positive x direction. The pressure gradient is 60 kPa/m, and it is negative. Compute the velocity at various points across the channel and show the results on a plot. Assume laminar flow.

87 Consider a steady, laminar flow through a straight horizontal tube having the constant elliptical cross section given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The streamlines are all straight and parallel. Investigate the possibility of using an equation for the z component of velocity of the form

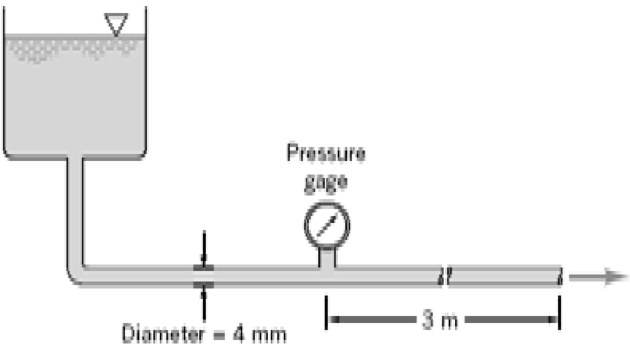
$$w = A\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

as an exact solution to this problem. With this velocity distribution, what is the relationship between the pressure gradient along the tube and the volume flowrate through the tube?

88 A fluid is initially at rest between two horizontal, infinite, parallel plates. A constant pressure gradient in a direction parallel to the plates is suddenly applied and the fluid starts to move. Determine the appropriate differential equation(s), initial condition, and boundary conditions that govern this type of flow. You need not solve the equation(s).

89 It is known that the velocity distribution for steady, laminar flow in circular tubes (either horizontal or vertical) is parabolic. (See Video V6.6.) Consider a 10-mm diameter horizontal tube through which ethyl alcohol is flowing with a steady mean velocity 0.15 m/s. (a) Would you expect the velocity distribution to be parabolic in this case? Explain. (b) What is the pressure drop per unit length along the tube?

90 A simple flow system to be used for steady flow tests consists of a constant head tank connected to a length of 4-mm-diameter tubing as shown in Fig. P 90. The liquid has a viscosity of 0.015 N · s/m², a density of 1200 kg/m³, and discharges into the atmosphere with a mean velocity of 2 m/s. (a) Verify that the flow will be laminar. (b) The flow is fully developed in the last 3 m of the tube. What is the pressure at the pressure gage? (c) What is the magnitude of the wall shearing stress, τ_{rz}, in the fully developed region?



■ FIGURE P 90

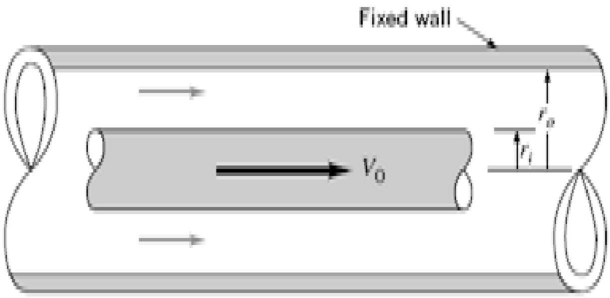
91 A highly viscous Newtonian liquid (ρ = 1300 kg/m³; μ = 6.0 N · s/m²) is contained in a long, vertical, 150-mm-diameter tube. Initially the liquid is at rest but when a valve at the bottom of the tube is opened flow commences. Although the flow is slowly changing with time, at any instant the velocity distribution is parabolic, that is, the flow is quasi-steady. (See Video V6.6.) Some measurements show that the average velocity, V, is changing in accordance with the equation $V = 0.1\ t$, with V in m/s when t is in seconds. (a) Show on a plot the velocity distribution (v_z vs. r) at t = 2 s, where v_z is the velocity and r is the radius from the center of the tube. (b) Verify that the flow is laminar at this instant.

92 (a) Show that for Poiseuille flow in a tube of radius R the magnitude of the wall shearing stress, τ_{rz}, can be obtained from the relationship

$$|(\tau_{rz})_{\text{wall}}| = \frac{4\mu Q}{\pi R^3}$$

for a Newtonian fluid of viscosity μ. The volume rate of flow is Q. (b) Determine the magnitude of the wall shearing stress for a fluid having a viscosity of 0.003 N · s/m² flowing with an average velocity of 100 mm/s in a 2-mm-diameter tube.

93 An incompressible Newtonian fluid flows steadily between two infinitely long, concentric cylinders as shown in Fig. P6.93. The outer cylinder is fixed, but the inner cylinder moves with a longitudinal velocity V_0 as shown. For what value of V_0 will the drag on the inner cylinder be zero? Assume that the flow is laminar, axisymmetric, and fully developed.



■ FIGURE P 93

94 An infinitely long, solid, vertical cylinder of radius R is located in an infinite mass of an incompressible fluid. Start with the Navier-Stokes equation in the θ direction and derive an expression for the velocity distribution for the steady flow case in which the cylinder is rotating about a fixed axis with a constant angular velocity ω. You need not consider body forces. Assume that the flow is axisymmetric and the fluid is at rest at infinity.

95 A viscous fluid is contained between two infinitely long, vertical, concentric cylinders. The outer cylinder has a radius r_o and rotates with an angular velocity ω. The inner cylinder is fixed and has a radius r_i. Make use of the Navier–Stokes equations to obtain an exact solution for the velocity distribution in the gap. Assume that the flow in the gap is axisymmetric (neither velocity nor pressure are functions of angular position θ within the gap) and that there are no velocity components other than the tangential component. The only body force is the weight.

96 For flow between concentric cylinders, with the outer cylinder rotating at an angular velocity ω and the inner cylinder fixed, it is commonly assumed that the tangential velocity (v_θ) distribution in the gap between the cylinders is linear. Based on the exact solution to this problem (see Problem 95) the velocity distribution in the gap is not linear. For an outer cylinder with radius r_o = 2.00 in. and an inner cylinder with radius r_i = 1.80 in., show, with the aid of a plot, how the dimensionless velocity distribution, $v_\theta/r_o\omega$, varies with the dimensionless radial position, r/r_o, for the exact and approximate solutions.

97 A viscous liquid ($\mu = 0.012\ \text{lb} \cdot \text{s/ft}^2$, $\rho = 1.79\ \text{slugs/ft}^3$) flows through the annular space between two horizontal, fixed, concentric cylinders. If the radius of the inner cylinder is 1.5 in. and the radius of the outer cylinder is 2.5 in., what is the pressure drop along the axis of the annulus per foot when the volume flowrate is $0.14\ \text{ft}^3/\text{s}$?

98 Plot the velocity profile for the fluid flowing in the annular space described in Problem P .97. Determine from the plot the radius at which the maximum velocity occurs and compare with the value predicted from Eq. **2.45** .

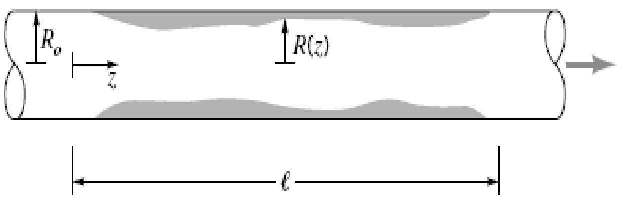
99 As is shown by Eq. **2.38** the pressure gradient for laminar flow through a tube of constant radius is given by the expression

$$\frac{\partial p}{\partial z} = -\frac{8\mu Q}{\pi R^4}$$

For a tube whose radius is changing very gradually, such as the one illustrated in Fig. P 99, it is expected that this equation can be used to approximate the pressure change along the tube if the actual radius, $R(z)$, is used at each cross section. The following measurements were obtained along a particular tube.

z/ℓ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$R(z)/R_o$	1.00	0.73	0.67	0.65	0.67	0.80	0.80	0.71	0.73	0.77	1.00

Compare the pressure drop over the length ℓ for this nonuniform tube with one having the constant radius R_o. *Hint*: To solve this problem you will need to numerically integrate the equation for the pressure gradient given above.



■ FIGURE P 99

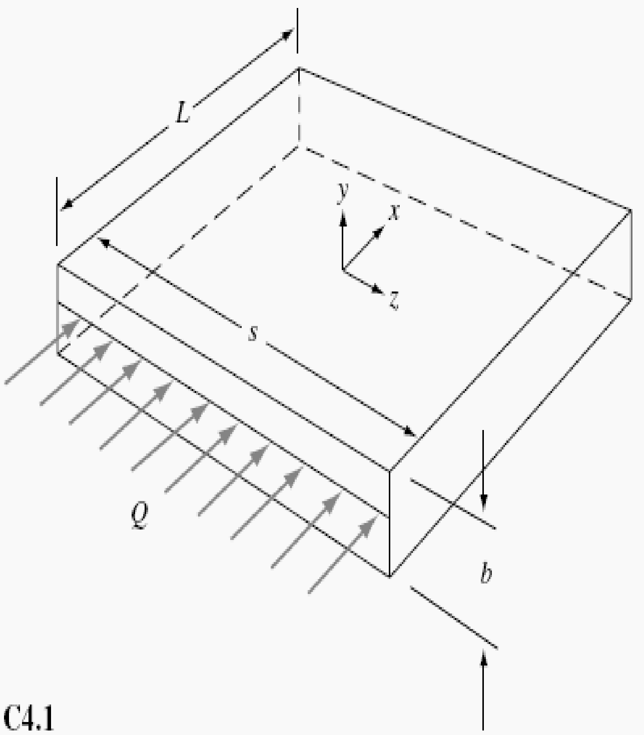
100 Show how Eq. **2.43** is obtained.

101 A wire of diameter d is stretched along the centerline of a pipe of diameter D. For a given pressure drop per unit length of pipe, by how much does the presence of the wire reduce the flowrate if (**a**) $d/D = 0.1$; (**b**) $d/D = 0.01$?

Comprehensive Problem

C4.1 In a certain medical application, water at room temperature and pressure flows through a rectangular channel of length $L = 10$ cm, width $s = 1.0$ cm, and gap thickness $b = 0.30$ mm as in Fig. C4.1. The volume flow rate is sinusoidal with amplitude $\hat{Q} = 0.50$ mL/s and frequency $f = 20$ Hz, i.e., $Q = \hat{Q} \sin(2\pi f t)$.

(*a*) Calculate the maximum Reynolds number ($\text{Re} = Vb/\nu$) based on maximum average velocity and gap thickness. Channel flow like this remains laminar for Re less than about 2000. If Re is greater than about 2000, the flow will be turbulent. Is this flow laminar or turbulent? (*b*) In this problem, the frequency is low enough that at any given time, the flow can be solved as if it were steady at the given flow rate. (This is called a quasi-steady assumption.) At any arbitrary instant of time, find an expression for streamwise velocity u as a function of y, μ, dp/dx, and b, where dp/dx is the pressure gradient required to push the flow through the channel at volume flow rate Q. In addition, estimate the maximum magnitude of velocity component u. (*c*) At any instant of time, find a relationship between volume flow rate Q and pressure gradient dp/dx. Your answer should be given as an expression for Q as a function of dp/dx, s, b, and viscosity μ. (*d*) Estimate the wall shear stress, τ_w as a function of $\hat{Q}$, f, μ, b, s, and time (t). (*e*) Finally, for the numbers given in the problem statement, estimate the amplitude of the wall shear stress, $\hat{\tau}_w$, in N/m^2.



C4.1

End of Part-2