ACCURACY, ERROR, AND UNCERTAINTY ANALYSIS

MEASUREMENT ERROR

All measurements have measurement errors. These errors are the differences between the measurements and the true value defined by the National Bureau of Standards (NBS). Uncertainty is the maximum error which might reasonably be expected and is a measure of accuracy, i.e., the closeness of the measurement to the true value. Measurement error has two components: a fixed error and a random error.

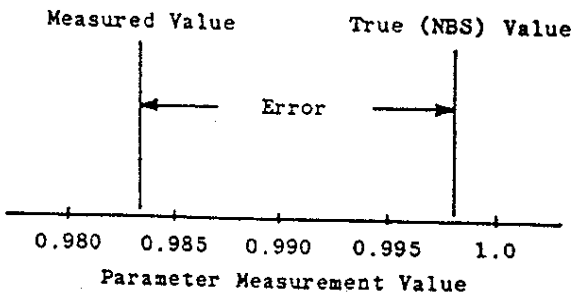


Fig. I-1 Measurement Error

Precision (Random Error)

Random error is seen in repeated measurements. Measurements do not and are not expected to agree exactly. There are always numerous small effects which cause disagreements. The variation between repeated measurements is called precision error. The standard deviation (σ) is used as a measure of the precision error. A large standard deviation means large scatter in the measurements. The statistic (s) is calculated to estimate the standard deviation and is called the precision index

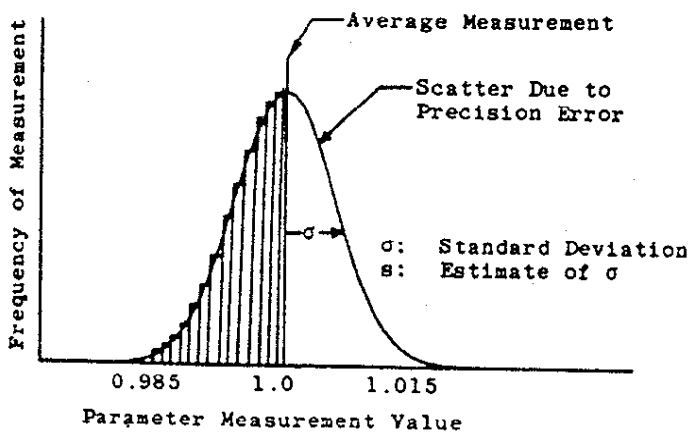


Fig. I-2 Precision Error

$$s = \sqrt{\frac{\sum_{i=1}^{N} (X_i - \bar{X})^2}{N - 1}}$$

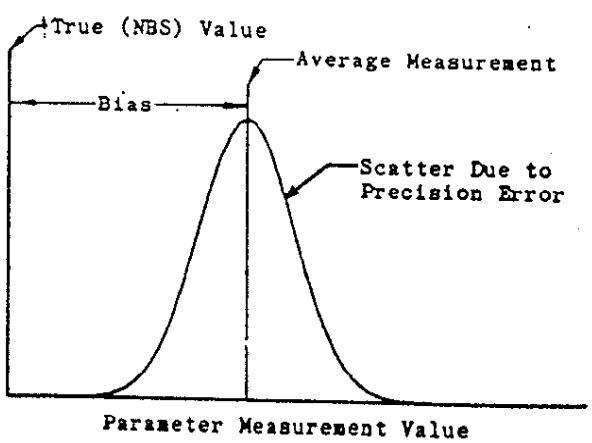
$\nu = N-1$ because $\bar{X}$ is the only calc. parameter

where N is the number of measurements made and $\bar{X}$ is the average value of individual measurements X_i.

Bias (Fixed Error)

The second component, bias, is the constant or systematic error. In repeated measurements, each measurement has the same bias. The bias cannot be determined unless the measurements are compared with the true value of the quantity measured.

Bias is categorized into five classes: (1) large known biases, (2) small known biases, (3) large unknown biases, and small unknown biases which may have (4) unknown sign (±) or (5) known sign



Large Known Biases

The large known biases are eliminated by comparing the instrument with a standard instrument and obtaining a correction. This process is called calibration.

Small Known Biases

Small known biases may or may not be corrected depending on the difficulty of the correction and the magnitude of the bias.

Large Unknown Biases

Unknown biases are not correctable. That is, they may exist, but the magnitude of the bias is not known, and perhaps even the sign is not known.

Every effort must be made to eliminate all large unknown biases. The introduction of such errors converts the controlled measurement process into an uncontrolled worthless effort. Large unknown biases usually come from human errors in data processing, incorrect handling and installation of instrumentation, and unexpected environmental disturbances such as shock and bad flow profiles. In a well-controlled measurement process, the assumption is that there are no large unknown biases. To ensure that a controlled measurement process exists, all measurements should be monitored with statistical quality control charts. A list of references describing the use of statistical quality control charts is included at the end of this section. Drifts, trends, and movements leading to out-of-control situations should be identified and investigated. Histories of data from calibrations are required for effective control. It is assumed throughout this Handbook that these precautions are observed and that the measurement process is in control; if not, the methods contained herein are invalid.

Small Biases, Unknown Sign, and Unknown Magnitude

In most cases, the bias error is equally likely to be plus or minus about the measurement. That is, it is not known if the limit is positive or negative, and the estimate reflects this. The bias limit is estimated as an upper limit on the maximum fixed error. For example, ±5 pounds is a typical bias limit.

It is both difficult and frustrating to estimate the limit of an unknown bias. To determine the exact bias in a measurement, it would be necessary to compare the true value and the measurements. This is almost always impossible. An effort must be made to obtain special tests or data that will provide bias information. The following are examples of such data:

1. Interlab, interfacility, intercompany tests on measurement devices, test rigs, and full-scale engines.

2. Flight test data versus altitude test chamber data versus ground test data.

3. Special comparisons of standards with instruments in the actual test environment.

4. Ancillary or concomitant functions that provide the same performance parameter; i.e., in an altitude engine test, airflow may be measured with (1) an orifice and (2) a bellmouth, (3) estimated from compressor speed-flow rig data, (4) estimated from turbine flow parameter, and (5) jet nozzle calibrations.

5. When it is known that a bias results from a particular cause, special calibrations may be performed allowing the cause to perturbate through its complete range to determine the range of bias.

If there is no source of data for bias, the judgment of the most knowledgeable instrumentation expert on the measurement must be used. However, without data, the upper limit on the largest possible bias error must reflect the lack of knowledge.

Small Biases, Known Sign, and Unknown Magnitude

Sometimes the physics of the measurement system provide knowledge of the sign but not the magnitude of the bias. For example, thermocouples radiate and conduct energy to indicate lower temperatures. The bias limits which result are nonsymmetrical, i.e., not of the form ±b. They are of the form $^{+b}_{-c}$ where both limits may be positive or negative or the limits may be of mixed sign as indicated. Table I below lists several nonsymmetrical bias limits for illustration.

Table I Nonsymmetrical Bias Limits

Bias Limits	Explanation
0, +10 deg	The bias will range from zero to plus 10 deg.
-5, +15 lb	The bias will range from minus 5 to plus 15 lb.
+3, +7 psia	The bias will range from plus 3 to plus 7 psia.
-8, -3 deg	The bias will range from minus 8 to minus 3 deg.

In summary, measurement systems are subject to two types of errors, bias and precision error (Fig. I-5). One sample standard deviation is used as the precision index. The bias limit is estimated as an upper limit on the maximum fixed error.

MEASUREMENT ERROR SOURCES

For purposes of illustration, the elemental error sources for the force measurement system will be treated in this section. These error sources fall into three categories:

1. Calibration Hierarchy Errors (2.2.1)
2. Data Acquisition Errors (2.2.2)
3. Data Reduction Errors (2.2.3)

Elemental error sources for other measurements will be enumerated in the section dealing with each measurement.

Calibration Hierarchy Errors

To demonstrate traceability of measurements to the NBS, whose standards are by definition the "truth," it is necessary to establish calibration hierarchies. Each level in the hierarchy, including NBS, constitutes an error source which contributes to the error in the final measurement. Calibration of all measurement instruments at the NBS is possible;

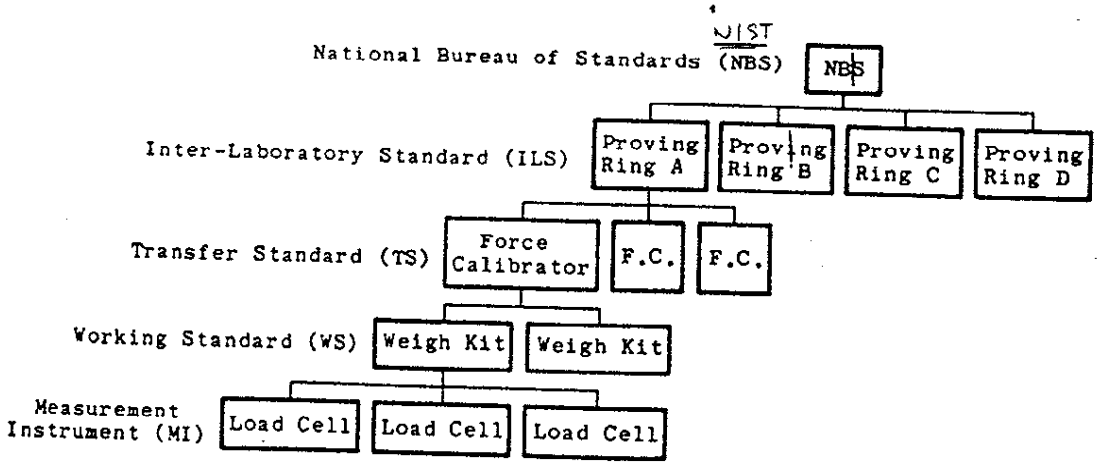


Fig. II-1 Force Measurement Calibration Hierarchy

however, such calibrations would be inconvenient, time consuming, and very expensive. The purpose here is to illustrate a typical hierarchy and to enumerate the error sources within. Figure II-1 is a typical force transducer calibration hierarchy. Associated with each comparison in the calibration hierarchy is a pair of elemental errors. These errors are the unknown bias and the precision index in each process. Note that these elemental errors are independent, e.g., b_{21} is not a function of b_{11}. The error sources are listed in Table V.

Table V Calibration Hierarchy Error Sources

Calibration	Bias Limit	Precision Index	Degrees of Freedom
NBS – ILS	b_{11}	s_{11}	df_{11}
ILS – TS	b_{21}	s_{21}	df_{21}
TS – WS	b_{31}	s_{31}	df_{31}
WS–MI	b_{41}	s_{41}	df_{41}

Data Acquisition Errors

Data are acquired by measuring the electrical output resulting from force applied to a strain-gage-type force measurement instrument. Figure II-2 illustrates some of the error sources associated with data acquisition. Other error sources such as electrical simulation, thrust bed mechanics, and environmental effects are also present. The best method to determine the effects of all of these error sources is to perform end-to-end calibrations and compare known applied forces with measured values. However, it is not always possible or even desirable to do this, and if this is the case, it is necessary to evaluate each of the elemental errors and combine them to determine the overall error.

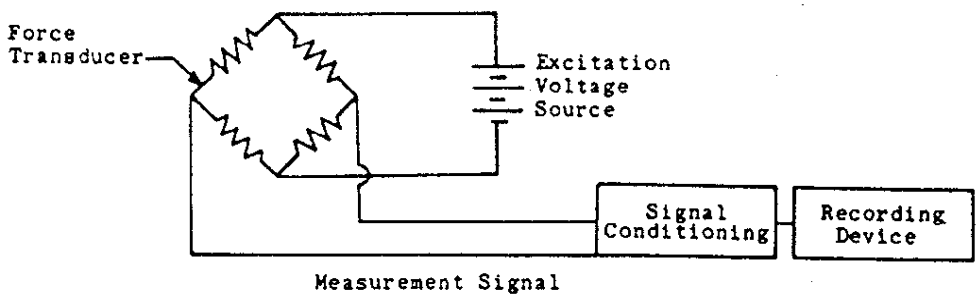


Fig. II-2 Data Acquisition System

All the data acquisition error sources are listed in Table VI. Symbols for the elemental bias and precision errors and for the degrees of freedom are shown.

Data Reduction Errors

Computers operate on raw data to produce output in engineering units. The errors in this process stem from calibration curve fits (Fig. II-3) and computer resolution.

Symbols for the data reduction error sources are listed in Table VII. These errors are often negligible in each process.

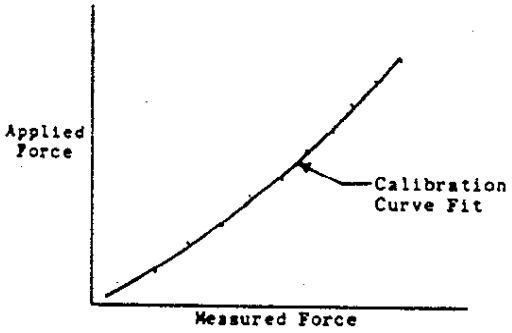


Fig. II-3 Calibration Curve

Table VI Data Acquisition Error Sources

Error Source	Bias Limit	Precision Index	Degrees of Freedom
Excitation Voltage	b_{12}	s_{12}	df_{12}
Electrical Simulation	b_{22}	s_{22}	df_{22}
Signal Conditioning	b_{32}	s_{32}	df_{32}
Recording Device	b_{42}	s_{42}	df_{42}
Force Transducer	b_{52}	s_{52}	df_{52}
Thrust Bed Mechanics	b_{62}	s_{62}	df_{62}
Environmental Effects	b_{72}	s_{72}	df_{72}

Table VII Data Reduction Error Sources

Error Source	Bias Limit	Precision Index	Degrees of Freedom
Calibration Curve Fit	b_{13}	s_{13}	df_{13}
Computer Resolution	b_{23}	s_{23}	df_{23}

MEASUREMENT UNCERTAINTY

For simplicity of presentation a single number (some combination of bias and precision) is needed to express a reasonable limit for error. The single number must have a simple interpretation (the largest error reasonably expected) and be useful without complex explanation. It is impossible to define a single rigorous statistic because the bias is an upper limit based on judgment which has unknown characteristics. Any function of these two numbers must be a hybrid combination of an unknown quantity (bias) and a statistic (precision). However, the need for a single number to measure error is so great that the adoption of an arbitrary standard is warranted. The standard most widely used is the bias limit plus a multiple of the precision index. This method is recognized and recommended by the NBS[2] and has been widely used in industry.

Uncertainty (Fig. I-6) may be centered about the measurement and is defined herein as:

$$U = \pm(B + t_{95}S) \qquad \text{(I-1)}$$

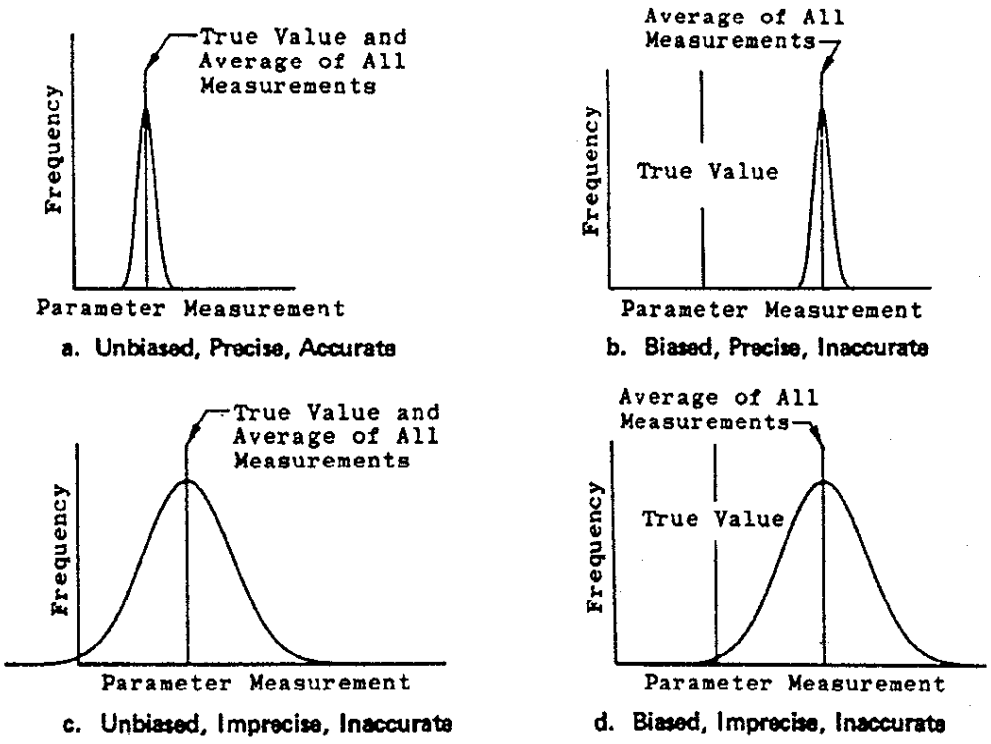


Fig. I-5 Measurement Error (Bias, Precision, and Accuracy)

where B is the bias limit, S is the precision index, and t_{95} is the 95th percentile point for the two-tailed Students "t" distribution (Table E-1, Appendix E). The t value is a function of the number of degrees of freedom (γ) used in calculating S. For small samples, t will be large, and for larger samples t will be smaller, approaching 1.96 as a lower limit. The use of the t arbitrarily inflates the limit U to reduce the risk of underestimating S when a small sample is used to calculate S. Since 30 degrees of freedom yield a t of 2.04 and infinite degrees of freedom yield a t of 1.96, an arbitrary selection of t = 2 for values of df from 30 to infinity was made, i.e., $U = \pm(B + 2S)$, when $df \geq 30$.

In a sample, the number of degrees of freedom is the size of the sample. When a statistic is calculated from the sample, the degrees of freedom associated with the statistic are reduced by one for every estimated parameter used in calculating the statistic. For example, from a sample of size N, $\overline{X}$ is calculated:

$$\overline{X} = \sum_{i=1}^{N} X_i / N \tag{I-2}$$

which has N degrees of freedom and

$$S = \sqrt{\frac{\sum_{i=1}^{N} (X_i - \overline{X})^2}{N-1}} \tag{I-3}$$

which has N-1 degrees of freedom because $\overline{X}$ (based on the same sample of data) is used to calculate S. In calculating other statistics, more than one degree of freedom may be lost. For example, in calculating the standard error of a curve fit, the number of degrees of freedom which are lost is equal to the number of estimated coefficients for the curve.

It is recommended that the uncertainty parameter (U) be used for simplicity of presentation; however, the components (bias, precision, and degrees of freedom) should be available in an appendix or in supporting documentation. These three components may be required (1) to substantiate and explain the uncertainty value, (2) to provide a sound technical base for improved measurements, and (3) to propagate the uncertainty from measured parameters to performance parameters, and from performance parameters to other more complex performance parameters (i.e. fuel flow to Thrust Specific Fuel Consumption (TSFC), TSFC to aircraft range, etc.). Although uncertainty is not a statistical confidence interval, it is an arbitrary substitute which is probably best interpreted as the largest error expected. Under any reasonable assumption for the distribution of bias, the coverage of U is greater than 95 percent, but this cannot be proved as the distribution of bias is both unknown and unknowable.

If there is a nonsymmetrical bias limit (Fig. I-7), the uncertainty U is no longer symmetrical about the measurement. The upper limit of the interval is defined by the upper limit of the bias interval (B^+). The lower limit is defined by the lower limit of the bias interval (B^-).

The uncertainty interval U is $U^- = B^- - t_{95}S$ to $U^+ = B^+ + t_{95}S$.

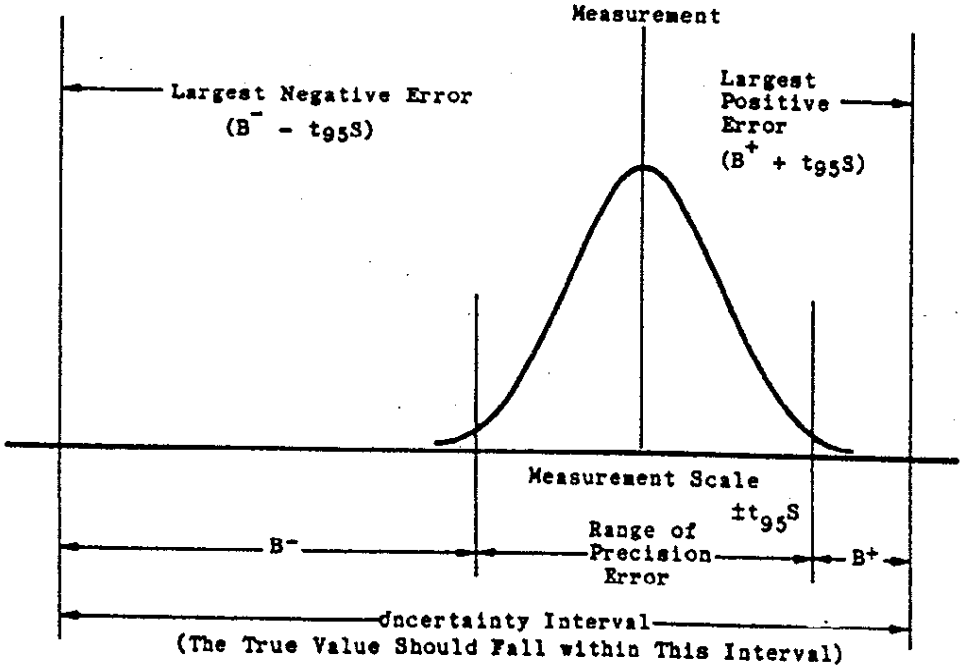


Fig. I-7 Measurement Uncertainty, Nonsymmetrical Bias

TABLE 4.1 Sample of Variable x

i	x_i	i	x_i
1	0.98	11	1.02
2	1.07	12	1.26
3	0.86	13	1.08
4	1.16	14	1.02
5	0.96	15	0.94
6	0.68	16	1.11
7	1.34	17	0.99
8	1.04	18	0.78
9	1.21	19	1.06
10	0.86	20	0.96

EXAMPLE 4.1

Compute the histogram and frequency distribution for the data of Table 4.1.

KNOWN

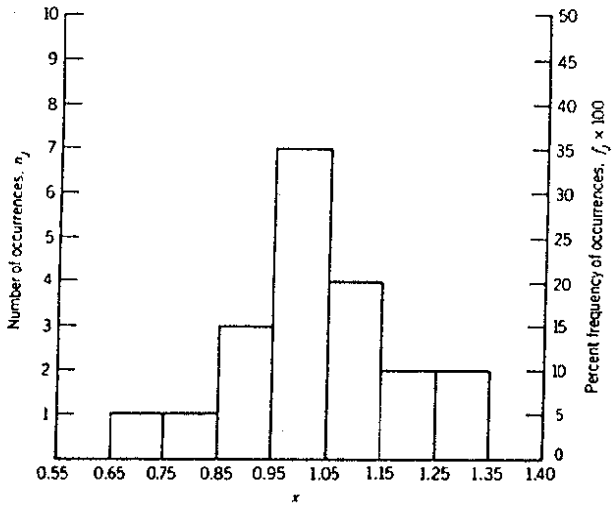
Data of Table 4.1

$N = 20$

ASSUMPTIONS

Fixed operating conditions

FIGURE 4.2 Histogram and frequency distribution for data in Table 4.1.



FIND

Histogram and frequency distribution

SOLUTION

To develop the histogram, compute a reasonable number of intervals for this data set. For $N = 20$, a convenient estimate of K is found from (4.2) to be

$$K = 1.87(N-1)^{0.40} + 1 = 7$$

Next, determine the maximum and minimum values of the data set and divide this range into K intervals. For a minimum of 0.68 and a maximum of 1.34, a $\delta x = 0.05$ is chosen. The intervals are shown below.

j	Interval	n_j	$f_j = n_j/N$
1	$0.65 \le x_i < 0.75$	1	0.05
2	$0.75 \le x_i < 0.85$	1	0.05
3	$0.85 \le x_i < 0.95$	3	0.15
4	$0.95 \le x_i < 1.05$	7	0.35
5	$1.05 \le x_i < 1.15$	4	0.20
6	$1.15 \le x_i < 1.25$	2	0.10
7	$1.25 \le x_i < 1.35$	2	0.10

Since at least one interval has an $n_j \ge 5$, the interval number is adequate. The results are plotted in Figure 4.2. The plot displays a definite central tendency at the maximum frequency of occurrence within the interval 0.95 to 1.05.

COMMENT

The sum of the number of occurrences,

$$\sum_{j=1}^{K} n_j,$$

must equal the total number of measurements, N. Likewise, the area under the percent frequency distribution curve will always equal the total frequency of occurrence of 100%, that is,

$$100 \times \sum_{j=1}^{K} f_j = 100\%.$$

EXAMPLE 4.4

Consider the data of Table 4.1. (a) Compute the sample statistics for this data set. (b) Estimate the interval of values over which 95% of the measurements of the measurand should be expected to lie. (c) Estimate the true mean value of the measurand at 95% probability based on this finite data set.

KNOWN

Table 4.1

$N = 20$

ASSUMPTIONS

Data set follows a normal distribution

FIND

$\bar{x}$, $\bar{x} \pm tS_x$ and $\bar{x} \pm tS_{\bar{x}}$

SOLUTION

The sample mean value is computed for the $N = 20$ values by the relation

$$\bar{x} = \frac{1}{20} \sum_{i=1}^{20} x_i = 1.02$$

This, in turn, is used to compute the sample standard deviation

$$S_x = \sqrt{\frac{1}{19} \sum_{i=1}^{20} (x_i - 1.02)^2} = 0.16$$

The degrees of freedom in the standard deviation are $\nu = N - 1 = 19$. From Table 4.4 at 95% probability, $t_{19,95}$ is 2.093. Then, the interval of values in which 95% of the measurements of x should lie is given by equation 4.15:

$$x_i = \bar{x} \pm (2.09 \times 0.16) = 1.02 \pm 0.33 \quad (95\%)$$

Accordingly, if a 21st data point were to be taken, there is a 95% probability that its value would lie between 0.69 and 1.35.

The true mean value is estimated by the sample mean value. However, the precision interval for this estimate is $\pm t_{19,95} S_{\bar{x}}$, where

$$S_{\bar{x}} = \frac{S_x}{N^{1/2}} = \frac{0.16}{(20)^{1/2}} = 0.04$$

Then, from equation 4.17

$$x' = \bar{x} \pm t_{19,95} S_{\bar{x}} = 1.02 \pm 0.08 \quad (95\%) \qquad (4.18)$$

TABLE 4.4 Student t Distribution

ν = Degrees of Freedom	t_{50}	t_{80}	t_{95}	t_{99}
1	1.000	6.314	12.706	63.657
2	0.816	2.920	4.303	9.925
3	0.765	2.353	3.182	5.841
4	0.741	2.132	2.770	4.604
5	0.727	2.015	2.571	4.032
6	0.718	1.943	2.447	3.707
7	0.711	1.895	2.365	3.499
8	0.706	1.860	2.306	3.355
9	0.703	1.833	2.262	3.250
10	0.700	1.812	2.228	3.169
11	0.697	1.796	2.201	3.106
12	0.695	1.782	2.179	3.055
13	0.694	1.771	2.160	3.012
14	0.692	1.761	2.145	2.977
15	0.691	1.753	2.131	2.947
16	0.690	1.746	2.120	2.921
17	0.689	1.740	2.110	2.898
18	0.688	1.734	2.101	2.878
19	0.688	1.729	2.093	2.861
20	0.687	1.725	2.086	2.845
21	0.686	1.721	2.080	2.831
30	0.683	1.697	2.042	2.750
40	0.681	1.684	2.021	2.704
50	0.680	1.679	2.010	2.679
60	0.679	1.671	2.000	2.660
∞	0.674	1.645	1.960	2.576

Least-Squares Regression Analysis

The regression analysis for a single variable of the form $y = f(x)$ provides an mth-order polynomial fit of the data in the form

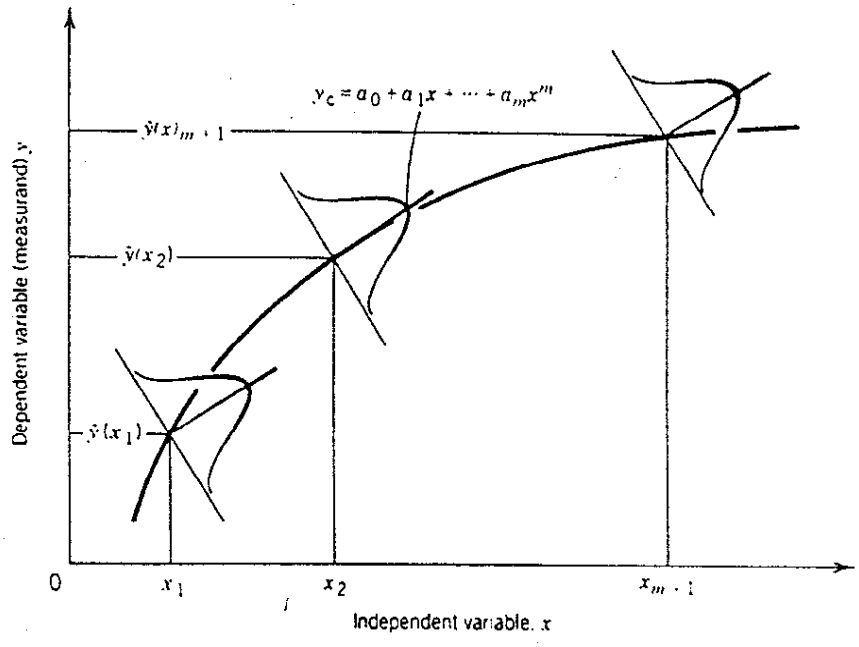
$$y_c = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m \tag{4.30}$$

where y_c refers to the value of the dependent variable obtained directly from the polynomial equation for a given value of x. For n different values of the independent variable included in the analysis, the highest order, m, of the polynomial that can be determined is restricted to $m \leq n - 1$. The values of the m coefficients $a_0, a_1, \ldots, a_m$ are determined by the analysis. The most common form for regression analysis for engineering applications is the *method of least-squares*. The least-squares technique attempts to minimize the sum of the squares of the deviations between the actual data and the polynomial fit of a stated order by adjusting the values of the coefficients, as necessary.

An mth-order polynomial relationship is to be found for a set of N data points of the form (x,y) in which x and y are the independent and dependent variables, respectively. Consider the situation in which N values of y exist, y_i, where $i = 1, 2, \ldots, N$, over n values of x. The task is to find the $m + 1$ coefficients, $a_0, a_1, \ldots, a_m$, of the polynomial of (4.30). Define the deviation between any dependent variable y_i and the polynomial as $y_i - y_{ci}$ where y_{ci} is the value of the polynomial evaluated at the data point (x_i, y_i). The sum of the squares of this deviation for all values of y_i, $i = 1, 2, \ldots, N$, is

$$D = \sum_{i=1}^{N} (y_i - y_{ci})^2 \tag{4.31}$$

FIGURE 4.9 Distribution of measured value y about each fixed value of independent variable x. The curve y_c represents a possible functional relationship.



The goal is to reduce D to a minimum for a given order of polynomial.

Combining equations 4.30 and 4.31, one can write

$$D = \sum_{i=1}^{N} [y_i - (a_0 + a_1 x + \cdots + a_m x^m)]^2 \tag{4.32}$$

Now the total differential of D is dependent on the $m + 1$ coefficients through

$$dD = \frac{\partial D}{\partial a_0} da_0 + \frac{\partial D}{\partial a_1} da_1 + \cdots + \frac{\partial D}{\partial a_m} da_m$$

To minimize the sum of the squares of the deviations, one wants dD to be zero. This is accomplished by setting each of the partial derivatives equal to zero:

$$\begin{aligned}
\frac{\partial D}{\partial a_0} = 0 &= \frac{\partial}{\partial a_0} \left\{ \sum_{i=1}^{N} [y_i - (a_0 + a_1 x + \cdots + a_m x^m]^2 \right\} \\
\frac{\partial D}{\partial a_1} = 0 &= \frac{\partial}{\partial a_1} \left\{ \sum_{i=1}^{N} [y_i - (a_0 + a_1 x + \cdots + a_m x^m]^2 \right\} \\
&\vdots \qquad\qquad \vdots \\
\frac{\partial D}{\partial a_m} = 0 &= \frac{\partial}{\partial a_m} \left\{ \sum_{i=1}^{N} [y_i - (a_0 + a_1 x + \cdots + a_m x^m)]^2 \right\}
\end{aligned} \tag{4.33}$$

This yields $m + 1$ equations which are solved simultaneously to yield the unknown regression coefficients, $a_0, a_1, \ldots, a_m$.

In general, the polynomial found using a regression analysis will not fit through every data point (x_i, y_i) exactly. Associated with any order of polynomial curve fit through a given set of data points, there will exist some deviation between the data point and the polynomial. One can compute a standard deviation based on the deviation of each data point and the fit by

$$S_{yx} = \sqrt{\frac{\sum_{i=1}^{N} (y_i - y_{ci})^2}{\nu}} \tag{4.34}$$

$\nu = N - 2$ for linear fit because a_0, a_1 calculated

where v is the degrees of freedom of the fit, $v = N - (m + 1)$. S_{yx} is referred to as the *standard error of the fit* and is a measure of the precision with which a polynomial describes the behavior of the data set.

The best order of polynomial fit to a particular data set is that lowest order of fit which reduces S_{yx} to an acceptable value *and* maintains a logical physical sense between dependent and independent variables. This latter point is important. If the underlying physics of a problem implies that a certain order relationship should exist between dependent and independent variables, there is no sense in forcing the data to fit any other order of polynomial regardless of the value of S_{yx}.

In engineering, the independent variable is often a known and controlled value. This is particularly true during calibration. In such cases, we can assume that the variance of the curve fit line is not due to the independent value. We can state the curve fit with its precision interval as

$$y_c \pm t_{v,P} S_{yx} \qquad (P\%) \tag{4.35}$$

where y_c is defined by (4.30).

On the other hand, if we consider variability in both the independent and dependent variables, then the curve fit and its precision interval are estimated by [1,4]

$$y_c + t_{v,P} S_{yx} \left[\frac{1}{N} + \frac{(x_i - \bar{x})^2}{\sum_{i=1}^{N} (x_i - \bar{x})^2} \right]^{1/2} \qquad (P\%) \tag{4.36}$$

This is the case if, say, we wished to compare the dietary fat intake of males with their blood cholesterol levels. If a random sample of the male population were to be taken, we would have no control over the independent variable, in this case, the number of individuals having a certain fat intake level. Because there would generally be fewer data points at the extreme ends of the range, the precision error at the extremes of the curve fit would increase. This is reflected in (4.36).

There is no rule that can be used to estimate which order fit will yield an acceptable value of S_{yx} without trial and error. This is the attractive feature of having a least-squares software package available. The choice of the actual order of fit used is always a compromise between the precision needed and the convenience of using a low-order polynomial.

EXAMPLE 4.8

The following data are suspected to follow a linear relationship. Find an appropriate equation of the first-order form.

x[cm]	y[V]
1.0	1.2
2.0	1.9
3.0	3.2
4.0	4.1
5.0	5.3

KNOWN

Independent variable, x
Dependent measured variable, y
$N = 5$

ASSUMPTIONS

Linear relation

FIND

$y_c = a_0 + a_1 x$

SOLUTION

We seek a polynomial of the form $y_c = a_0 + a_1 x$, which minimizes the term $D = \sum_{i=1}^{N} (y_i - y_{c_i})^2$:

$$\frac{\partial D}{\partial a_0} = 0 = -2\sum_{i=1}^{N} [y_i - (a_0 + a_1 x_i)]$$

$$\frac{\partial D}{\partial a_1} = 0 = -2\sum_{i=1}^{N} [y_i - (a_0 + a_1 x_i)] x_i$$

yielding

$$\sum_{i=1}^{N} [y_i - (a_0 + a_1 x_i)] = 0$$

$$\sum_{i=1}^{N} [y_i - (a_0 + a_1 x_i)] x_i = 0$$

Solving simultaneously for the coefficients a_0 and a_1 yields

$$a_0 = \frac{\Sigma x_i \Sigma x_i y_i - \Sigma x_i^2 \Sigma y_i}{(\Sigma x_i)^2 - N \Sigma x_i^2}$$

$$a_1 = \frac{\Sigma x_i \Sigma y_i - N \Sigma x_i y_i}{(\Sigma x_i)^2 - N \Sigma x_i^2} \tag{4.37}$$

From the data set, one finds from (4.37) $a_0 = 0.02$ and $a_1 = 1.04$. Hence,

$$y_c = 0.02 + 1.04x \quad \text{V}$$

COMMENT

Although the polynomial described by y_c is the linear curve fit for this data set, we still have no idea of how good this curve fits this data set or even if a first-order fit is appropriate. This is studied below.

Linear Polynomials

For linear polynomials a correlation coefficient, r, can be found by

$$r = \sqrt{1 - \frac{S_{yx}^2}{S_y^2}} \tag{4.38}$$

where

$$S_y^2 = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{y})^2$$

The correlation coefficient represents a quantitative measure of the linear association between x and y. It is bounded by ± 1 which represents perfect correlation; the sign indicates that y increases or decreases with x. For $\pm 0.9 < r \leq \pm 1$, a linear regression can be considered as a reliable relation between y and x. Alternatively, the value r^2 is often reported, which is indicative of how well the variance in y is accounted for by the fit. It is a ratio of the variation assumed by the linear fit to the actual measured variations in the data. However, the correlation coefficient and the r^2 value are only indicators of the hypothesis that y and x are linearly related. They are not effective precision indicators of y_c. The S_{yx} value is used for that purpose.

The precision estimate of the slope of the fit can be estimated by

$$S_{a_1} = S_{y,x}\sqrt{\frac{N}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}} \tag{4.39}$$

For example, S_{a_1} would provide a measure of the static sensitivity error of a measurement system based on a linear fit of the calibration data.

The precision estimate of the zero intercept can be estimated by

$$S_{a_0} = S_{y,x}\sqrt{\frac{N\sum_{i=1}^{N} x_i^2}{N\left[N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2\right]}} \tag{4.40}$$

An error in a_0 would offset a calibration curve from its y intercept. The derivation and further discussion on equations 4.38–4.40 can be found in [1,2,4].

EXAMPLE 4.9

Compute the correlation coefficient and the standard error of the fit for the data in Example 4.8.

KNOWN

$y_c = 0.02 + 1.04x$ V

ASSUMPTIONS

Errors are normally distributed

FIND

r and $S_{y,x}$

SOLUTION

Direct application of equation 4.38 with the data set yields for the correlation coefficient $r = 0.996$. An equivalent estimator is r^2. Here $r^2 = 0.99$, which indicates that 99% of the variance in y is accounted for by the fit, whereas only 1% is unaccountable. These values suggest that a linear fit is a reliable relation between x and y. The precision error between the data and this fit can be quantified through $S_{y,x}$. Using equation 4.36, $S_{y,x} = 0.16$ with degrees of freedom, $\nu = N - (m + 1) = 3$.

COMMENT

The t estimator, $t_{3,95} = 3.18$, establishes a precision or confidence interval about the fit of $\pm(t_{3,95}S_{y,x}) = \pm 0.50$. Accordingly, the polynomial fit can be stated at 95% confidence as

$$y_c = 1.04x + 0.02 \pm 0.50 \text{ V} \quad (95\%)$$

This curve is plotted in Figure 4.10 with its 95% confidence interval. The regression polynomial with its precision interval is the only acceptable way to report a curve fit to a data set.

Propagation of Basic Uncertainties into Results

Consider the second power equation where R is the result and x_1, x_2, etc. are the fundamental, independent, measured quantities. Then

$$U_R^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 U_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 U_{x_2}^2 + \ldots\ldots + \left(\frac{\partial f}{\partial x_n}\right)^2 U_{x_n}^2$$

where U_R is the absolute uncertainty and

$$u_R^2 = \frac{U^2}{R^2} = \frac{\left(\frac{\partial f}{\partial x_1}\right)^2 U_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 U_{x_2}^2 + \ldots\ldots + \left(\frac{\partial f}{\partial x_n}\right)^2 U_{x_n}^2}{R^2}$$

where u_R is the relative uncertainty which could be expressed as a percentage if multiplied by 100, and R is the magnitude of U_R.

This second power equation can be used for both random and systematic uncertainties. The random uncertainty U_{ran} and the systematic uncertainty B can be combined to give the total uncertainty U_{tot} using the equation

$$U_{tot} = \sqrt{U_{ran}^2 + B^2}\ .$$

Applications

The above calculation of the uncertainty can be used to:

1. determine (or estimate) the accuracy of test results.
2. aid to plan an experiment and to select instrumentation by determining the effects of component uncertainties on the results.
3. show where improvements or an increased expense will be most worthwhile.
4. shows where less accurate (e.g. control room) instruments may be acceptable.
5. show which uncertainties may be reduced by replication.

Example

Consider a pitot-static tube and manometer to measure the velocity of an air jet traveling at a steady speed. A pitot-static tube is often used with a manometer.

The following table shows the various readings and their absolute uncertainty.

PARAMETER	INSTRUMENT	READING	UNCERTAINTY	SOURCE
T_a	Liquid-in-glass thermometer	67°F	0.5°F	Instrument specified
Δp	Manometer	8 in. H_2O	0.1 in. H_2O	Operator observation of column "bounce"
p_a	Aneroid Barometer	14.7 psia	0.3 psi	Instrument specified

Note: The uncertainties includes both systematic and random components.

The basic equation for the result (velocity) is:

$$V = \sqrt{\frac{2\,\Delta p}{\rho}}$$

where Δp is the difference between the total pressure and the static pressure (i.e. the velocity pressure). The density ρ is not an independent measurement but is obtained from the data and the ideal gas law

$$\rho = \frac{p_a}{R\,T_a}$$

where T_a and p_a are the (absolute) static temperature and (absolute) static pressure of the fluid stream respectively. The equation for the fluid velocity is then

$$V = C\sqrt{\frac{2\,\Delta p\,R\,T_a}{p_a}}$$

The constant C is normally 1.0 for a pitot-static tube if the fluid velocity is relatively low (below a Mach number of 0.2), no turbulence, fully developed flow, and perfect alignment of the pitot-static tube with the fluid velocity. The second power equation gives

$$U_V^2 = \left(\frac{\partial V}{\partial C}\right)^2 U_C^2 + \left(\frac{\partial V}{\partial \Delta p}\right)^2 U_{\Delta p}^2 + \left(\frac{\partial V}{\partial T_a}\right)^2 U_{T_a}^2 + \left(\frac{\partial V}{\partial p_a}\right)^2 U_{p_a}^2$$

where the gas constant R has no uncertainty. Performing the derivative operations, the calculating equation for V is then

$$U_V^2 = \left(\frac{2\ \Delta p\ R\ T_a}{p_s}\right) U_C^2 + \frac{1}{4}\left(\frac{2\ R\ T_a}{\Delta p\ p_a}\right) U_{\Delta p}^2 + \frac{1}{4}\left(\frac{2\ \Delta p\ R\ T_a}{p_a^3}\right) U_{p_a}^2 + \frac{1}{4}\left(\frac{2\ \Delta p\ R}{p_a\ T_a}\right) U_{T_a}^2 .$$

C is a calibration coefficient for the Pitot-static tube and taken as 1.0. However, C may differ from 1.0 due to: (a) compressibility, (b) turbulence, © the velocity gradient, and (d) misalignment. Dividing by

$$V^2 = C\left(\frac{2\ \Delta p\ R\ T_a}{p_a}\right)^2$$

gives

$$\frac{U_V^2}{V^2} = \frac{U_C^2}{C^2} + \frac{1}{4}\frac{U_{\Delta p}^2}{(\Delta p)^2} + \frac{1}{4}\frac{U_{p_a}^2}{p_a^2} + \frac{1}{4}\frac{U_{T_a}^2}{T_a^2}$$

or

$$u_V^2 = u_C^2 + \frac{1}{4}u_{\Delta p}^2 + \frac{1}{4}u_{p_a}^2 + \frac{1}{4}u_{T_a}^2.$$

For a pitot-static tube, $u_C =$ 0.01 and the numerical values give

$$u_V = \left[0.0001 + \frac{1}{4}(0.00016) + \frac{1}{4}(0.00042) + \frac{1}{4}(0.0000009)\right]^{\frac{1}{2}} = 0.0156$$

or 1.56%. Some possible improvements in the uncertainty would occur if:

1. The static-pitot tube had better calibration or $U_C = 0.005$. ? most important
2. The static pressure measurement were more accurate or $U_{p_a} = \pm 0.1$ psia.
3. The thermocouple were more accurate or $U_{T_a} = \pm 0.1°F$.

The proper method for combining elemental measurement uncertainty values is to determine the root-sum-square values of the elemental bias limits and the elemental precision indices separately. Then, apply the uncertainty formula to the combined bias limits and precision indices. In some cases, the same value will be obtained if the uncertainties are root-sum-squared directly. However, this is not a general rule, and large errors in the combined uncertainty (10 to 25 percent) can result. Further, the root-sum-squared uncertainty value will be smaller (optimistic) than the proper uncertainty estimate, and the estimate is a significant underestimate of the true measurement error.

For example, in combining the following uncertainties the root-sum-square of the uncertainties was 18.38 units. The correct value was 23.21 units.

Bias Limit (B)	Precision Index (S)	Uncertainty
1	6	±13
11	1	±13

where Uncertainty = $\pm(B + t_{95}S)$. $t_{95} \approx 2$

Now the bias limit for the combined parameter is the root-sum-square of 1 and 11:

$$B = \sqrt{1^2 + 11^2} = \sqrt{122} = 11.05$$

The precision index for the combined parameter is the root-sum-square of 6 and 1:

$$S = \sqrt{6^2 + 1^2} = \sqrt{37} = 6.08$$

The Uncertainty is thus:

$$U = \pm(B + 2S) = \pm[11.05 + 2(6.08)] = \pm 23.21$$

The root-sum-square of the original uncertainties is

$$\sqrt{(13)^2 + (13)^2} = \sqrt{169 + 169} = \sqrt{338} = 18.38$$

Now,

$$\frac{23.21 - 18.38}{18.38} \times 100 = 26.3\%$$

and over 25 percent error has been introduced just because of the wrong propagation of error formula.

REPORTING ERROR

The definition of the components, bias limit, precision index, and the limit (U) suggests a format for reporting measurement error. The format will describe the components of error, which are necessary to estimate further propagation of the errors, and a single value (U) which is the largest error expected from the combined errors. Additional information, degrees of freedom for the estimate of S, is required to use the precision index. These numbers provide all the information necessary to describe and use the measurement error. The reporting format is:

1. S, the estimate of the precision index, calculated from data.

2. ν, the degrees of freedom associated with the estimate of the precision index (S).

3. B, the upper limit of the bias error of the measurement process or B^- and B^+ if the bias limit is nonsymmetrical.

4. $U = \pm(B + t_{95} S)$, the uncertainty limit, beyond which measurement errors would not reasonably fall. The t value is the 95th percentile of the two-tailed Student "t" distribution.

5. U, the interval between $U^- = B^- - t_{95} S$ and $U^+ = B^+ + t_{95} S$. These limits should be reported when the bias limit is nonsymmetrical.

The model components, S, ν, B, and U, are required to report the error of any measurement process. As recommended in Section 1.4, for simplification, the first three components may be relegated to the detailed sections of uncertainty reports and presentations. The first three components, S, ν, and B, are necessary to propagate the errors further, to propagate the uncertainty to more complex parameters, and to substantiate the uncertainty limit.